Thermal state of the harmonic oscillator (ref.: Gerry & Knight, Ch. 2.5)

According to statistical mechanics, at thermal equilibrium the probability P_n that the n^{th} level is excited is:

$$P_n = \frac{\exp\left(-\frac{E_n}{k_B T}\right)}{\sum_k \exp\left(-\frac{E_k}{k_B T}\right)} \quad \text{where} \quad E_n = n\hbar\omega \quad *$$

$$= \frac{1}{Z} \exp\left(-\frac{E_n}{k_B T}\right) \quad \text{with} \quad Z = \frac{1}{1 - \exp\left(\frac{-\hbar\omega}{k_B T}\right)}$$

* (we discard the zero point energy since it has no impact on the final density matrix)

- The density operator is: $\hat{\rho}_{th} = \sum_{n=0}^{\infty} P_n |n\rangle\langle n|$

$$= \frac{1}{Z} \sum_n \tilde{p}^n |n\rangle\langle n|$$

with $\tilde{p} = e^{-\frac{\hbar\omega}{k_B T}} < 1$

- <u>Mean occupancy</u> (phonon number):

$$\bar{n}_{th} = \langle \hat{n} \rangle = \mathrm{Tr}(\hat{n}\hat{\rho}_{th}) = \sum_{n=0}^{\infty} \langle n | \hat{n}\hat{\rho}_{th} | n\rangle$$

$$= \sum_n n P_n = \frac{1}{Z} \sum_n n\tilde{p}^n \quad \rightarrow \tilde{p} = e^{-x}$$

↳ Geometric series:

$$\sum_n n e^{-nx} = -\frac{d}{dx} \sum_{n=0}^{\infty} e^{-nx} = -\frac{d}{dx}\left(\frac{1}{1-e^{-x}}\right) = \frac{e^{-x}}{(1-e^{-x})^2}$$

Thermal state of the harmonic oscillator

Therefore with $\frac{1}{Z} = 1 - e^{-x}$ we obtain:

$$\bar{n}_{th} = \frac{e^{-x}}{1-e^{-x}} = \frac{\tilde{p}}{1-\tilde{p}} \quad \text{with} \quad \tilde{p} = e^{-\frac{\hbar\omega}{k_B T}}$$

we can rewrite: $\boxed{\bar{n} = \left(e^{\frac{\hbar\omega}{k_B T}} - 1\right)^{-1}}$

This is the Bose-Einstein occupancy

<u>Limiting cases</u>:

- $\hbar\omega \ll k_B T \longrightarrow \tilde{p} \approx 1 - \frac{\hbar\omega}{k_B T} \longrightarrow \bar{n}_{th} \approx \frac{k_B T}{\hbar\omega} \gg 1$
- $\hbar\omega \gg k_B T \longrightarrow \tilde{p} \ll 1 \longrightarrow \bar{n}_{th} \approx e^{-\frac{\hbar\omega}{k_B T}} \ll 1$

↳ Show slide with curves.

Since $\bar{n} = \frac{1}{\tilde{p}^{-1} - 1} \Rightarrow \tilde{p}^{-1} = \frac{1}{\bar{n}} + 1 = \frac{\bar{n}+1}{\bar{n}} \Rightarrow \boxed{\tilde{p} = \frac{\bar{n}}{\bar{n}+1}}$

Then $\frac{1}{Z} = 1 - \tilde{p} = \frac{1}{\bar{n}+1}$ and the density operator writes:

$$\boxed{\hat{\rho}_{th} = \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle\langle n|} = (1-\tilde{p}) \sum_{n=0}^{\infty} p^n |n\rangle\langle n|$$

and $\boxed{P_n = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}}$

Intensity fluctuations: $\langle(\Delta n)^2\rangle = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2$

It can be shown that: $\langle\hat{n}^2\rangle = \bar{n} + 2\bar{n}^2$

So that: $\langle(\Delta n)^2\rangle = \bar{n} + \bar{n}^2$

Thermal state of the harmonic oscillator

In particular we can compute the intensity correlation function $g^{(2)} \equiv \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2} = \frac{\langle a^\dagger (a a^\dagger - 1) a\rangle}{\langle \hat{n}\rangle^2} = \frac{\langle \hat{n}^2 - \hat{n}\rangle}{\langle \hat{n}\rangle^2}$

$$= \frac{2\bar{n}^2}{\bar{n}^2} = 2$$

Rem. Consider measuring $g^{(2)}$ on two thermal modes:

$$I = I_a + I_b \rightarrow g^{(2)} = \frac{\langle : I^2 : \rangle}{\langle I\rangle^2} = \frac{\langle : I_a^2 + I_b^2 + 2 I_a I_b : \rangle}{4\langle I_a\rangle^2}$$

$$= \frac{g_a^{(2)}}{2} + \frac{1}{2} = 1 + 1/2$$

For N modes $g_N^{(2)} = 1 + 1/N$

Walls & Milburn Ch. 4 p. 58:

- The thermal density operator with $P_n = \frac{1}{1+\bar{n}}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n$ can be obtained by maximizing the entropy:

$S = -\mathrm{Tr}\,(\rho \ln \rho)$ subject to the constraint $\mathrm{Tr}\{\hat{\rho}\hat{a}^\dagger\hat{a}\} = \bar{n}$

- General expression for $g^{(2)}(0) = 1 + \frac{V(n) - \bar{n}}{\bar{n}^2}$ where $V(n)$ is the variance $V(n) = \langle (\Delta n)^2\rangle$

For thermal state $V(n) = \bar{n}^2 + \bar{n} \Rightarrow g^{(2)}(0) = 2$

coherent state $V(n) = \bar{n} \Rightarrow g^{(2)}(0) = 1$

Interaction of light with a collection of Raman-active oscillators

Ref.: T. von Foerster and R. J. Glauber, Phys. Rev. A 3 1484 (1971)

Model: • Plane waves polarized along z propagating along $\pm x$

• Linear chain of diatomic molecules, displacement $u_z(x,t)$

The induced polarisation for a given field depends on $u(x,t)$:

$$\vec{P} = \varepsilon_0 \underbrace{\chi}_{\text{susceptibility}}(u(x,t))\, \vec{E}_{tot} \quad \rightarrow \text{we expand: } \chi(x,t) = \chi_0 + \underbrace{\chi_R}_{\text{Raman susceptibility}} u(x,t) + \mathcal{O}(u^2)$$

where $\chi_R = \frac{\partial \chi}{\partial u}\Big|_{u=0}$

Electric Displacement: $\vec{D} = \varepsilon_0 \vec{E}(x,t) + \vec{P}$

$$= \varepsilon_0 (1+\chi) E_z + \varepsilon_0 \chi_R E_z u_z$$

<u>NB</u>: in general χ_R is a tensor: χ_{ijk} and the last term

$$D_i^{(R)} = \varepsilon_0 \chi_{ijk} E_j u_k$$

<u>Energy density of the system</u>:

$$\mathcal{H}_{tot} = \frac{1}{2}\vec{D}.\vec{E} + \frac{1}{2}\vec{B}.\vec{B} = \mathcal{H}_{free} + \mathcal{H}_{int}$$

with $\boxed{\mathcal{H}_{int} = \frac{1}{2}\varepsilon_0 \chi_R E_z^2(x,t)\, u_z(x,t)}$

Now we decompose the field in a strongly populates plane wave (laser) and the rest: $\vec{E}(x,t) \longrightarrow \mathcal{E}_L(x,t) + \vec{E}(x,t)$

With this:

$$\mathcal{H}_{int} = \underbrace{\frac{1}{2}\varepsilon_0 \chi_R \mathcal{E}_L^2(x,t)\, u(x,t)}_{\text{Shift of zero point motion of the molecular vibration}} + \underbrace{\varepsilon_0 \chi_R \mathcal{E}_L(x,t) E_z(x,t)\, u(x,t)}_{\text{Important term}} + \underbrace{\frac{1}{2}\varepsilon_0 \chi_R E_z^2(x,t)\, u(x,t)}_{\text{Second order effect}}$$

Interaction of light with a collection of Raman-active oscillators

Expansion of the fields :

- $\mathcal{E}_L(x,t) = \mathcal{E}_L \cos(k_L x - \omega_L t) = \frac{\mathcal{E}_L}{2} e^{i(k_L x - \omega_L t)} + \frac{\mathcal{E}_L}{2} e^{-i(k_L x - \omega_L t)}$

$$= \underset{\sim e^{-i\omega_L t}}{\mathcal{E}_L^{(+)}} + \underset{\sim e^{+i\omega_L t}}{\mathcal{E}_L^{(-)}} \quad \text{(analytical signals)}$$

- $E_z(x,t) = \sum_k \sqrt{\frac{\hbar \omega_k}{2L\varepsilon_0}} \left(i\hat{a}_k e^{i(kx - \omega_k t)} + h.c. \right) = E^{(+)} + E^{(-)}$

- $u(x,t) = \sum_q \sqrt{\frac{\hbar}{2L\rho\Omega}} \left(\hat{b}_q e^{i(qx - \Omega t)} + h.c \right) = u^{(+)} + u^{(-)}$

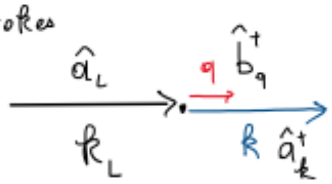
This yields 8 terms in the Hamiltonian, but those with $\mathcal{E}^{(-)}E^{(-)}$ and $\mathcal{E}^{(+)}E^{(+)}$ oscillates around $2\omega_L \to$ cannot conserve energy.

Rmq : conservation of energy $\Leftrightarrow$ time-independent Hamiltonian.
$\hookrightarrow$ fast oscillating terms average to zero.

We finally get : $\mathcal{H}_{int} = \mathcal{H}_S + \mathcal{H}_A$ with $\mathcal{H}_S = \frac{\varepsilon_0}{2}\chi_R \mathcal{E}_L^{(+)} E^{(-)} u^{(-)} + h.c.$

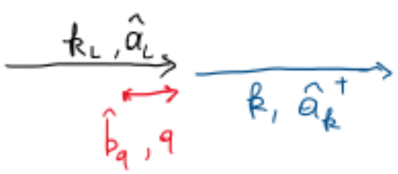
$$\mathcal{H}_A = \frac{\varepsilon_0}{2}\chi_R \mathcal{E}^{(+)} E^{(-)} u^{(+)} + h.c.$$

Stokes



$$\begin{cases} q_S + k_S = k_L \\ \Omega + \omega_S = \omega_L \end{cases}$$

Anti-stokes :



$$\begin{cases} k_L + q_A = k_A \\ \omega_L + \Omega = \omega_A \end{cases}$$

* We get two new modes populated at $\omega_L \pm \Omega$

* They couple to the same phonon mode if $n \simeq c^{ste}$:

$$q_S = k_L - k_S \quad \& \quad q_A = k_A - k_L$$

Interaction of light with a collection of Raman-active oscillators

<u>If constant index</u> : $k = \frac{\omega}{nc} \rightarrow q_s = \frac{\Omega}{nc} = q_A$

<u>Rem</u> : Since $\lambda \gg d$ where d is lattice constant, we have $\Delta k \sim \frac{2\pi}{\lambda} \ll \frac{\pi}{d} \sim$ size of first Brillouin zone.

<u>Conclusions</u> : The laser field couples two ~~new~~ EM modes (S, A) to the same phonon mode.

The interaction Hamiltonian for the Stokes field is :

$$\hat{H}_s = \int \mathcal{H}_s \, dx = \int \frac{dx}{L} \frac{\varepsilon_0}{2} \chi_R \frac{\xi_L}{2} \frac{\hbar}{2} \sqrt{\frac{\omega_s}{\varepsilon_0 \rho \Omega}} \, i \hat{a}_s^\dagger \hat{b}^\dagger e^{i(0-0)} + h.c.$$

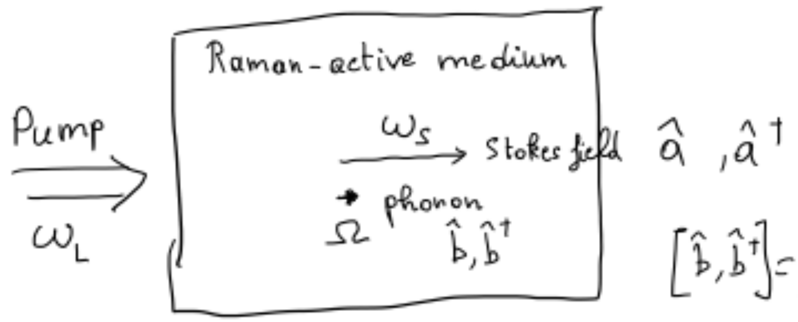
$$= \frac{1}{8} \sqrt{\varepsilon_0} \chi_R \xi_L \sqrt{\frac{\omega_s}{\rho \Omega}} \, \hbar \left(i \hat{a}_s^\dagger \hat{b}^\dagger - i \hat{a}_s \hat{b} \right)$$

$$\boxed{\hat{H}_s = i\hbar \left(g_s \hat{a}_s^\dagger \hat{b}^\dagger - g_s^* \hat{a}_s \hat{b} \right)}$$ two-mode squeezing

and $$\boxed{\hat{H}_A = i\hbar \left(g_A \hat{a}_A^\dagger \hat{b} - g_B^* \hat{a}_A \hat{b}^\dagger \right)}$$ beam-splitter

Two-mode squeezed vacuum

Situation :



$$[\hat{b}, \hat{b}^\dagger] = [\hat{a}, \hat{a}^\dagger] = 1$$

- Initially , modes a and b are in vacuum or weekly occupied thermal state (say $\bar{n}_{th} < 10^{-2}$) : $|\Psi_{in}\rangle_{a,b} = |0\rangle_a \otimes |0\rangle_b$

↳ the anti-Stokes interaction plays little role .

Stokes interaction Hamiltonian :

$$\hat{H}_S = i\hbar \left(g\, \hat{a}^\dagger \hat{b}^\dagger - g^* \hat{a}\hat{b} \right) = i\hbar \left(\lambda e^{i\theta}\, \hat{a}^\dagger \hat{b}^\dagger - \lambda e^{-i\theta}\, \hat{a}\hat{b} \right)$$

- We look for the Stokes / phonon state after an interaction time t :

$$|\Psi_t\rangle = \hat{U}(t)\, |\Psi_{in}\rangle \quad \text{where} \quad \hat{U}(t) = \exp\left(-\frac{it}{\hbar}\hat{H}_S\right)$$

(Schrödinger picture)

$$= \exp\left(g t\, \hat{a}^\dagger \hat{b}^\dagger - g^* t\, \hat{a}\hat{b} \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(g t\, \hat{a}^\dagger \hat{b}^\dagger - g^* t\, \hat{a}\hat{b} \right)^n \rightarrow \text{Hard!}$$

- In the Heisenberg picture we can write the Heisenberg-Langevin equations of motion :

$$i\hbar \frac{\partial \hat{a}}{\partial t} = [\hat{a}, \hat{H}_s] \Rightarrow \frac{\partial \hat{a}}{\partial t} = g\, [\hat{a}, \hat{a}^\dagger]\, \hat{b}^\dagger = g\, \hat{b}^\dagger = \lambda e^{i\theta}\, \hat{b}^\dagger \quad (1)$$

$$\frac{\partial \hat{b}}{\partial t} = g\, \hat{a}^\dagger = \lambda e^{i\theta}\, \hat{a}^\dagger \quad (2)$$

$$\hookrightarrow \frac{\partial \hat{b}^\dagger}{\partial t} = \lambda e^{-i\theta}\, \hat{a} \quad (3)$$

Two-mode squeezed vacuum

From (1) and (3) we get: $\frac{\partial^2 \hat{a}}{\partial t^2} = \lambda^2 \hat{a}(t)^2$

(Similarly: $\frac{\partial^2 \hat{b}}{\partial t^2} = \lambda^2 \hat{b}(t)^2$)

Solutions are of the form $\hat{a}(t) = \hat{A} e^{\lambda t} + \hat{B} e^{-\lambda t}$

Initial conditions: $t = 0 \to \hat{a}_0 = \hat{A} + \hat{B}$ (4)

Derivative:

$$\lambda e^{i\theta} \hat{b}_0^\dagger = \lambda \hat{A} - \lambda \hat{B}$$

$$\Leftrightarrow e^{i\theta} \hat{b}_0^\dagger = \hat{A} - \hat{B} \qquad (5)$$

From (4) and (5) we get: $\hat{A} = \frac{\hat{a}_0 + e^{i\theta}\hat{b}_0^\dagger}{2}$

$$\hat{B} = \frac{\hat{a}_0 - e^{i\theta}\hat{b}_0^\dagger}{2}$$

Therefore: $\hat{a}(t) = \hat{a}_0 \frac{e^{\lambda t} + e^{-\lambda t}}{2} + \hat{b}_0^\dagger e^{i\theta} \frac{e^{\lambda t} - e^{-\lambda t}}{2}$

$$\boxed{\hat{a}_t = \hat{a}_0 \cosh(\lambda t) + \hat{b}_0^\dagger e^{i\theta} \sinh(\lambda t)}$$

and $\hat{b}(t) = \frac{1}{\lambda e^{-i\theta}} \frac{\partial \hat{a}^\dagger}{\partial t} = e^{i\theta}\left(\hat{a}_0^\dagger \sinh(\lambda t) + \hat{b}_0 e^{-i\theta} \cosh(\lambda t)\right)$

$$\boxed{\hat{b}_t = \hat{b}_0 \cosh(\lambda t) + \hat{a}_0^\dagger e^{i\theta} \sinh(\lambda t)}$$

We have shown that $\hat{U}^\dagger(t)\, \hat{a}_0\, \hat{U}(t) = \hat{a}_t$

NB: this can also be done with the help of the Baker-Hausdorf lemma:

$$e^{x\hat{A}} \hat{B} e^{-x\hat{A}} = \hat{B} + x[\hat{A},\hat{B}] + \frac{x^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dots$$

$\Rightarrow$ Now, how to switch to Schrödinger picture?

Two-mode squeezed vacuum

Preliminary remarks: * $\hat{H}_S$ commute with $\hat{n}_a - \hat{n}_b$:

$$[\hat{H}_S, \hat{n}_a - \hat{n}_b] = [\hat{H}_S, \hat{n}_a] - [\hat{H}_S, \hat{n}_b] = 0$$

This means that $\langle \hat{n}_a \rangle = \langle \hat{n}_b \rangle$ (constant of motion)

* $\hat{H}_S$ creates and annihilates by pair a, b, therefore if the initial state is the vacuum, after time t:

$$|\Psi_t\rangle = \sum_{n=0}^{\infty} \alpha_n |n\rangle_a \otimes |n\rangle_b = \sum_n \alpha_n |n,n\rangle = |\xi\rangle$$

* For a weak interaction $t\lambda \ll 1$ we can expand:

$$\hat{U}(t) = 1 + t\lambda e^{i\theta} a^\dagger b^\dagger - t\lambda e^{-i\theta} ab + \frac{1}{2!}\left(t\lambda e^{i\theta} a^\dagger b^\dagger - t\lambda e^{-i\theta} ab\right)^2 + \dots$$

Applied on vacuum: $|\Psi_t\rangle \simeq |0,0\rangle + \lambda t e^{i\theta} |1,1\rangle + \frac{t^2\lambda^2 e^{2i\theta}}{2!} |2,2\rangle - \frac{\lambda^2 t^2}{2} |0,0\rangle$

$$\simeq \left(1 - \frac{t^2\lambda^2}{2}\right)|0,0\rangle + t\lambda e^{i\theta} |1,1\rangle + \frac{t^2\lambda^2 e^{2i\theta}}{2} |2,2\rangle$$

$$\simeq |0,0\rangle + \lambda t e^{i\theta} |1,1\rangle \quad \text{if } \lambda t \ll 1$$

Calculation of the wave function of the squeezed vacuum

We set $\mu = \cosh(\lambda t)$ and $\nu = \sinh(\lambda t) e^{i\theta}$

We use $\hat{a}_0 |0,0\rangle = 0$ to write $\hat{U}(t)\, \hat{a}_0 \underbrace{\hat{U}^\dagger \hat{U}}_{\hat{I}_d} |0,0\rangle = 0$

but $\hat{U}(t)|0,0\rangle = |\xi\rangle$ (squeezed vacuum)

and $\hat{U}(t)\hat{a}_0\hat{U}^\dagger(t)$ is the inverse transformation $\hat{a}_t \to \hat{a}_0$, that is:

$$\begin{cases} \hat{a}_t = \mu \hat{a}_0 + \nu \hat{b}_0^\dagger \\ \hat{b}_t = \mu \hat{b}_0 + \nu \hat{a}_0^\dagger \end{cases} \Rightarrow \quad (\mu^2 - \nu^2)\hat{a}_0 = \mu \hat{a}_t - \nu \hat{b}_t^\dagger = \hat{a}_0 \quad \text{and} \quad \mu \hat{b}_t - \nu \hat{a}_t^\dagger = \hat{b}_0$$

Therefore: $(\mu \hat{a}_t - \nu \hat{b}_t^\dagger)|\xi\rangle = 0$ With $|\xi\rangle = \sum_n \alpha_n |n,n\rangle$ we have:

$$\sum_n \alpha_n \left(\mu \sqrt{n}\, |n-1, n\rangle - \nu \sqrt{n+1}\, |n, n+1\rangle \right) = 0$$

This can only be true if each coefficient is zero:

$$\forall n, \quad \alpha_{n+1}\,\mu - \alpha_n\,\nu = 0 \iff \frac{\alpha_{n+1}}{\alpha_n} = \frac{\nu}{\mu}$$

thus: $\alpha_n = \alpha_0 \left(\frac{\nu}{\mu}\right)^n = \alpha_0\, e^{in\theta} \tanh(\lambda t)^n$

α_0 is determined from the normalisation condition $\sum |\alpha_n|^2 = 1$:

$$1 = \alpha_0^2 \sum_{n=0}^{\infty} \tanh^{2n}(\lambda t) = \alpha_0^2 \left(1 - \tanh^2(\lambda t)\right)^{-1}$$

$$\Rightarrow \alpha_0^2 = 1 - \tanh^2(\lambda t) = \frac{\cosh^2 - \sinh^2}{\cosh^2} = \frac{1}{\cosh^2(\lambda t)}$$

We finally arrive at the expression of the two-mode squeezed vacuum in the Fock basis:

$$\boxed{\begin{aligned} |\xi\rangle &= \frac{1}{\cosh(\lambda t)} \sum_{n=0}^{\infty} e^{in\theta} \tanh^n(\lambda t)\, |n,n\rangle \\ &= \sqrt{1 - |\tilde{p}|^2} \sum_n \tilde{p}^n\, |n,n\rangle \quad \text{with} \quad \tilde{p} = e^{i\theta} \tanh(\lambda t) \end{aligned}}$$

Two-mode squeezed vacuum

"two-photon coherent state"

Density operator:

$$\boxed{\hat{\rho}_s = |\xi\rangle\langle\xi| = (1 - |\tilde{p}|^2) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \tilde{p}^n \tilde{p}^{*m} |n,n\rangle\langle m,m|}$$

Two-mode squeezed vacuum

· Properties of the marginal states.

We trace over mode b:

$$\hat{\rho}_a = \frac{1}{Tr_{a,b}(\hat{\rho})} \cdot Tr_b(\hat{\rho}) = Tr_b(\hat{\rho}) = \sum_{m=0}^{\infty} {}_b\langle m|\hat{\rho}|m\rangle_b$$

$$= \sum_m (1 - |\tilde{p}|^2)\, \tilde{p}^m \tilde{p}^{*m}\, |m\rangle_a\langle m|$$

$$= (1-p) \sum_n p^n |n\rangle\langle n| \quad \text{with} \quad p = |\tilde{p}|^2$$

This is a thermal state with $p = \frac{\bar{n}}{1+\bar{n}}$ or $\bar{n} = \frac{p}{1-p}$; $p = e^{-\frac{\hbar\omega}{k_B T}}$

- Probality of creating one pair:

$$P(1,1) = (1-p)\,p$$

- n pair: $P(n,n) = (1-p)\,p^n$

Fock state heralding

We start from the two mode squeezed vacuum and take

$\tilde{p} \in \mathbb{R} \rightarrow \tilde{p} = \sqrt{p}$:

$$|\xi\rangle = \sqrt{1-p} \sum_{n=0}^{\infty} \sqrt{p}^{\,n} |n\rangle_a \otimes |n\rangle_b$$

If we detect at least one photon in mode a, mode b is described by the density matrix:

$$\hat{\rho}^{cond} = \frac{1}{K} \sum_{n=1}^{\infty} P(n,n)\, |n\rangle\langle n| = \frac{1}{K}(1-p) \sum_{n=1}^{\infty} p^n\, |n\rangle\langle n|$$

For normalisation: $K = (1-p)\sum_{n=1}^{\infty} p^n = (1-p)\left(\frac{1}{1-p} - 1\right) = p$

Finally: $\hat{\rho}_{cond} = \frac{1-p}{p} \sum_{n=1}^{\infty} p^n\, |n\rangle\langle n|$

for $p \ll 1$, $\boxed{\hat{\rho}_{cond} = |1\rangle\langle 1| + p|2\rangle\langle 2| + \mathcal{O}(p^2)}$

~ Fock state

Beam-splitter interaction

We start from the anti-Stokes interaction Hamiltonian:

$$\hat{H}_A = i\hbar\left(g\,\hat{a}^\dagger\hat{b} - g^*\,\hat{b}^\dagger\hat{a}\right) = i\hbar\left(\lambda e^{i\theta}\,\hat{a}^\dagger\hat{b} - \lambda e^{-i\theta}\,\hat{b}^\dagger\hat{a}\right)$$

Heisenberg equations:

$$\frac{\partial \hat{a}}{\partial t} = \frac{1}{i\hbar}[\hat{a},\hat{H}] = \lambda e^{i\theta}\hat{b} \quad \text{and} \quad \frac{\partial \hat{b}}{\partial t} = -\lambda e^{-i\theta}\,\hat{a}$$

$$\hookrightarrow \frac{\partial^2 \hat{a}}{\partial t^2} = -\lambda^2 \hat{a} \quad \text{and} \quad \frac{\partial^2 \hat{b}}{\partial t^2} = -\lambda^2 \hat{b}$$

Solutions of the form: $\hat{a}(t) = \hat{A}\cos(\lambda t) + \hat{B}\sin(\lambda t)$

$\underline{t=0}$: $\hat{a}_0 = \hat{A}$; $\left.\frac{\partial \hat{a}}{\partial t}\right|_{t=0} = \lambda \hat{B} = \lambda e^{i\theta}\hat{b}_0 \Rightarrow \hat{B} = e^{i\theta}\hat{b}_0$

Therefore:

$$\boxed{\begin{aligned} \hat{a}_t &= \hat{a}_0 \cos(\lambda t) + \hat{b}_0 \sin(\lambda t)\, e^{i\theta} \\ \hat{b}_t &= \hat{b}_0 \cos(\lambda t) - e^{-i\theta}\,\hat{a}_0 \sin(\lambda t) \end{aligned}}$$

Normally ordered expectation values, if $|\Psi_a\rangle = |0\rangle$:

$$\boxed{\langle :(a_t^\dagger a_t)^n: \rangle = \langle (a_t^\dagger)^n (a_t)^n \rangle = \sin^2(\lambda t)\, \langle :(b_0^\dagger b_0)^n: \rangle}$$

In particular: $\boxed{g^{(2)}_{a_t} = g^{(2)}_{b_0}}$ statistical properties of mode b_0 are mapped on mode a_t

Beam-splitter interaction

Example : input Fock state on mode b : $|\Psi_{in}\rangle = |0\rangle_a \otimes |1\rangle_b$

where $|1\rangle_b = \hat{b}_0^\dagger |0\rangle$

we have to express $\hat{b}_0$ in terms of $\hat{a}_t, \hat{b}_t$:

$$\cos(\lambda t)\,\hat{a}_t - \sin(\lambda t) e^{i\theta}\,\hat{b}_t = \hat{a}_0$$

$$e^{-i\theta}\sin(\lambda t)\,\hat{a}_t + \cos(\lambda t)\,\hat{b}_t = \hat{b}_0$$

Therefore $\hat{b}_0^\dagger |0,1\rangle_{a,b} = e^{+i\theta}\sin(\lambda t)|1,0\rangle + \cos(\lambda t)|0,1\rangle_{a_t, b_t}$

↳ The phonon $b_0^\dagger$ is mapped onto an anti Stokes photon $a_t^\dagger$ with probability $\sin^2(\lambda t)$

Input = 2 phonon Fock State $|\Psi_{in}\rangle = |0\rangle_a \otimes |2\rangle_b$

where $|2\rangle_{b_0} = \dfrac{\hat{b}_0^\dagger \hat{b}_0^\dagger}{\sqrt{2}} |0\rangle_{b_0}$

$$= \frac{1}{\sqrt{2}}\Big(\sqrt{2}\, e^{2i\theta}\sin^2(\lambda t)|2,0\rangle + 2e^{i\theta}\cos(\lambda t)\sin(\lambda t)|1,1\rangle + \sqrt{2}\cos^2(\lambda t)|0,2\rangle\Big)$$

More generally for a "n-phonon" Fock state input :

$$\boxed{|\Psi_{in}\rangle = |0,n\rangle \rightarrow |\Psi_{out}\rangle = \sum_{k=0}^{n} \binom{n}{k}^{1/2} \tau^k \rho^{n-k} |n-k, k\rangle}$$

where $\tau = \cos(\lambda t)$ and $\rho = e^{i\theta}\sin(\lambda t)$

Photons split like if they were distinguishable particle, and the proba to have k transmitted and $n-k$ reflected is $\binom{n}{k} |\tau|^{2k} |\rho|^{2(n-k)}$