Statistical mixtures in quantum mech.

System + environment always in pure state (assume)

\[ |\Psi\rangle = \sum_{j,k} c_{j,k} |\Psi_j\rangle \otimes |\Psi_k\rangle \]

This implies no description in terms of a "pure" quantum state of the system.

What is important in q.m. is measurement.

If \( O_s \) observable of the system, what is the exp. val. on \( |\Psi\rangle \)?

\[ \langle O_s \rangle = \langle \Psi | O_s | \Psi \rangle \]

\[ = \sum_{j,k} c_{j,k}^{*} c_{j,m} \langle \Psi_j | O_s | \Psi_j \rangle \langle \Psi_m | \Psi_m \rangle \]

\[ = \sum_{j,k} \sum_{i,m} c_{e,m}^{*} c_{j,m} \langle \Psi_e | O_s | \Psi_i \rangle \langle \Psi_i | \Psi_j \rangle \]

\[ = \sum_{j,k} P_{j,k} \langle O_{s_{j,k}} \rangle = \text{Tr} (\rho \; O_s) \]

\[ P_{j,k} = \sum_{m} c_{e,m}^{*} c_{j,m} \]

\[ = \sum_{m} \langle \Psi_e | \otimes | \Psi_m \rangle | \Psi_m \rangle \otimes | \Psi_j \rangle \]
\[ \hat{\rho} = \sum_{i=1}^{N_e} |\psi_i\rangle \langle \psi_i| \hat{\rho}_{ij} |\psi_i\rangle \langle \psi_i| \]

= \text{Tr}_B (|\psi\rangle \langle \psi|) \quad \text{partial trace}

\hat{\rho} \text{ operator} : \mathcal{H}_s \rightarrow \mathcal{H}_s

1\) \quad \hat{\rho}^+ = \hat{\rho}

2\) \quad \text{Tr}(\hat{\rho}) = \sum_{j=m} \sum_{j=m} |c_{jm}|^2 = 1

3\) \quad \langle \psi | \hat{\rho} | \psi \rangle \geq 0 \forall |\psi\rangle \in \mathcal{H}_s

Spectral decomposition.
\[ \hat{\rho} = \sum_{i} \rho_i |\psi_i\rangle \langle \psi_i| \quad \sum_{i} \rho_i = 1 \quad \rho_i \geq 0 \]

not unique if \( |\psi_i\rangle \) not orthogonal.

\[ \text{Tr} (\hat{\rho}^2) \leq \text{Tr} (\hat{\rho}) = 1 \]

If we have a pure state \(| \psi \rangle \) and an obs. \( O \), then
\[ \langle O \rangle = \langle \psi | O | \psi \rangle = \text{Tr} (O | \psi \rangle \langle \psi |) \]

Then, for a quantum state ("pure")
\[ \hat{\rho} = |\psi\rangle \langle \psi| \]
\[ \text{Tr} (\rho^2) = 1 \quad \text{pure state} \quad \text{in} \quad \text{concil.} \]

Define \( S(\mathcal{H}) \) Giouville space of d.m.

If \( \rho_1, \rho_2 \in S(\mathcal{H}) \)

\[ \rho = d \rho_1 + (1-d) \rho_2 \in S(\mathcal{H}) \quad d \in [0,1] \]

convex property of \( S(\mathcal{H}) \)

pure states live at the boundary of \( S(\mathcal{H}) \)

and cannot be expressed as convex linear comb. of nontrivial d.m.'s.

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The adv. of the d.m. formalism is that of unifying statistical mixtures and pure states.

\[ \rho = \sum_j \rho_j |\psi_j\rangle \langle \psi_j| \]

\[ \langle \psi \rangle = \text{Tr} (\rho \psi) = \sum_j \rho_j \langle \psi | 10 | \psi_j \rangle \]

\[ = \sum_j \rho_j \langle \psi | \rho \psi \rangle \]

from which the notion of statistical mixture becomes clear.
If two subsystems are not correlated, then
\[ \rho = \rho_1 \otimes \rho_2 \]
this means,
\[ \hat{A} = \hat{A}_1 \otimes \hat{A}_2 \]
\[ \langle \hat{A} \rangle = \text{Tr} (\hat{A} \rho) = \]
\[ = \text{Tr}_1 (A_1 \rho_1) \text{Tr}_2 (A_2 \rho_2) \]
\[ = \langle A_1 \rangle \langle A_2 \rangle \]

However, in general, if \[ \rho_i = \text{Tr}_j (\rho) \]
it is not true that
\[ \rho = \rho_1 \otimes \rho_2 \]
because there can be entanglement.

Notion of separability for D07
As the whole q. m. can be formulated in terms of density m., we derive now the time evolution equation for the d. m.

Closed quantum system.

\[ i \hbar \frac{d}{dt} \left| \psi(t) \right> = H(t) \left| \psi(t) \right> \]

\[ \left| \psi(t) \right> = U(t,0) \left| \psi(0) \right> \]

\[ i \frac{d}{dt} U(t,0) = H(t) U(t,0) \quad U(0,0) = I \]

If \( H(t) = H \Rightarrow U(t,0) = e^{-iHt/\hbar} \)

Otherwise

\[ U(t,0) = T \exp \left[ -i \int_{t_0}^{t} ds H(s) \right] \]

If at \( t = 0 \) \( \rho = \sum_j |\psi_j \rangle \langle \psi_j| \)

\[ \Rightarrow \rho(t) = \sum_j \rho_j(t) U(t,0) \left| \psi_j \right> \langle \psi_j | U^+(t,0) \]

\[ = U(t,0) \rho(0) U^+(t,0) \]

\[ \Rightarrow \frac{d}{dt} \rho(t) = -i \left[ H(t), \rho(t) \right] \]

Liouville–von Neumann equation
\[
\frac{d}{dt} \rho(t) = \mathcal{L}(t) \rho(t)
\]
\[
\mathcal{L}(t) \text{ Liouville super-operator}
\]

\[
\rho(t) = \mathcal{T} \exp \left[ \int_{t_0}^{t} ds \, \mathcal{L}(s) \right] \rho(t_0)
\]

For \( \mathcal{L}(t) = \mathcal{L} \)

\[
\rho(t) = \exp \left( \mathcal{L} t \right) \rho(0)
\]
We now wish to describe an open Q.S. S coupled to another system B the environment (unaming)
Total Hilbert space $\mathcal{H}_s \otimes \mathcal{H}_B$

$$H = \mathcal{H}_s \otimes I_B + \mathcal{H}_s \otimes \mathcal{H}_B + H_Z(t)$$

Why do so?
- $S + B$ too big to be modeled
- Interested only in observables of $S$
- $B$ not well known.

Observables of $S$ : $A \otimes I_B$

$$\langle A \rangle = \text{Tr}_s(A \rho_s)$$ partial trace

$$\rho_s = \text{Tr}_B(\rho)$$

Interest in knowing $\rho_s(t)$
\[ \rho_s(t) = \text{Tr}_B \left( \rho(t) \right) \]
\[ = \text{Tr}_B \left( U(t, t_0) \rho(t_0) U^\dagger(t, t_0) \right) \]

If \( S + B \) form a closed system, then
\[ \frac{d \rho_s(t)}{dt} = -i \text{Tr}_B \left( [H(t), \rho(t)] \right) \]
Quantum Markovian process

A Markovian environment is one for which the timescale associated to the correlations is much shorter than any timescale relevant to the system dynamics. This condition implies the absence of any memory effect for the reduced system dynamics.

A more formal definition of a quantum Markov process can be introduced in relation to the dynamical map.

At $t=0$ the full density matrix is assumed to be the uncorrelated system-environment state

$$\rho(0) = \rho_s(0) \otimes \rho_B$$

The time evolution is then

$$\rho_s(0) \rightarrow \rho_s(t) = \text{Tr}_B \left( \hat{O}(t,0) [\rho(0) \otimes \rho_B] \hat{O}^+(t,0) \right)$$

$$= V(t) \rho_s(0)$$

Here $V(t): S \rightarrow S$ is a map from the space of reduced density matrices into itself, called the dynamical map.
Then as a function of time $t$ we have a family of maps $\{V(t)\}_{t \geq 0}$ with $V(0) =$ identity.

The Markovian property is then translated into the semigroup property (in the homogeneous case)

$$V(t_1)V(t_2) = V(t_1 + t_2) \quad t_1, t_2 \geq 0$$

In other words, if $\rho_s(t_2) = V(t_2)\rho_s(0)$, then at $t = t_1 + t_2$ the state $\rho_s(t_1 + t_2)$ will only depend on $\rho_s(t_2)$ and not on its past history.

The quantum dynamical semigroup property is analogous to the Chapman-Kolmogorov equation holding for classical stochastic processes. Its homogeneous version is

$$\rho_{t + t'}(x | x) = \int dx'' \rho_t(x | x'') \rho_{t'}(x'' | x')$$

$\forall, t', t' > 0$

which has an intuitive interpretation.
Under general conditions, it is possible to write a quantum dynamical semigroup in terms of a generator $L$ as

$$V(t) = \exp(\mathcal{L} t)$$

which implies the differential equation

$$\frac{d}{dt} \rho_s(t) = \mathcal{L} \rho_s(t)$$

which is called the Markovian quantum master equation.

The generator $\mathcal{L}$ is a linear super-operator $\mathcal{L}: \mathcal{S} \rightarrow \mathcal{S}$ called Liouvillian.

For the case of an isolated system we have seen that

$$\mathcal{L} \rho(t) = -i [H, \rho(t)]$$

Highlight that this is a Lévy process as the Schrödinger equation can be solved from the initial condition only.

* **Bounded $\mathcal{L}$, separable (Metric) $\mathcal{H}$**

For most physical systems, $\mathcal{L}$ is not bounded (neither is $H$).
We now wish to determine the form of \( V(t) \) and of the Liouvillian \( \mathcal{L} \) from general principles, retaining only the Markov assumption. We introduce the spectral decomposition of the bath density matrix

\[
\rho_B = \sum_{\kappa} \lambda_\kappa |\varphi_\kappa\rangle \langle \varphi_\kappa| \quad \sum_{\kappa} \lambda_\kappa = 1 \quad \lambda_\kappa \geq 0
\]

Then

\[
V(t) \rho_S = \Tr_B \left( U(t,0) \rho_S \otimes \rho_B \ U^*(t,0) \right)
\]

\[
= \sum_{\kappa} \langle \varphi_\kappa | U(t,0) \rho_S \otimes \rho_B \ U^*(t,0) | \varphi_\kappa \rangle \rho_S \langle \varphi_\kappa | U(t,0) | \varphi_\kappa \rangle \sqrt{\lambda_\kappa}
\]

\[
= \sum_{\kappa \beta} W_{\kappa \beta}(t) \rho_S \ W_{\kappa \beta}^\dagger(t)
\]

\( W_{\kappa \beta}(t) : \mathcal{N}_S \rightarrow \mathcal{N}_B \)

\( W_{\kappa \beta}(t) = \sqrt{\lambda_\beta} \langle \varphi_\kappa | U(t,0) | \varphi_\beta \rangle \)

(Pay attention to the fact that \( U \) is an operator on \( \mathcal{N}_S \otimes \mathcal{N}_B \) and that taking matrix elements over both states \( |\varphi_\kappa\rangle \) renacts into an operator \( W : \mathcal{N}_S \rightarrow \mathcal{N}_B \)

We see that

\[
\sum_{\kappa \beta} W_{\kappa \beta}^\dagger(t) W_{\kappa \beta}(t) = \mathbb{1}
\]

Then

\[
\Tr_S \left( V(t) \rho_S \right) = \sum_{\kappa \beta} \Tr_S \left( W_{\kappa \beta} \rho_S W_{\kappa \beta}^\dagger \right)
\]

\[
= \Tr_S \left( \sum_{\kappa \beta} W_{\kappa \beta}^\dagger W_{\kappa \beta} \rho_S \right)
\]
Thus $V(t)$ is a linear, trace preserving map. It is also completely positive:

$$
\langle \psi | \sum_{\kappa \beta} \psi_{\kappa \beta} \rho_{\kappa} \psi_{\kappa \beta} | \psi \rangle = \sum_{\kappa \beta} \langle \psi_{\kappa \beta} | \rho_{\kappa} | \psi_{\kappa \beta} \rangle \geq 0 \text{ from the positivity of } \rho_{\kappa}
$$

To derive the general form of $L$, we assume a finite dimensional Hilbert space for the system, $\mathcal{H}$, with dimension $N$. The present considerations can be extended to infinite dimension in the case of countable states, assuming $L$ is bounded. More generally, a different strategy is adopted, namely to derive $L$ microscopically for a specific model, and realize that it has the general form that we are going to derive below.
Consider the $N^2$-dimensional space of operators on $\mathcal{H}$. Consider a set of orthonormal operators $\hat{F}_j$, $j = 1, 2, \ldots, N^2$ such that

$$(\hat{F}_j, \hat{F}_k) = \mathcal{T}_\mathcal{H} \left( \hat{F}_j \hat{F}_k^+ \right) = \delta_{jk}$$

(this scalar product is the one defining the trace distance).

One of the $\hat{F}_j$ is assumed proportional to $\mathbb{I}_\mathcal{H}$

$\hat{F}_{N^2} = \frac{1}{\sqrt{N}} \mathbb{I}_\mathcal{H}$. Then all other $\hat{F}_j$ are traceless

$\mathcal{T}_\mathcal{H}(\hat{F}_j) = 0 \quad j = 1, \ldots, N^2 - 1$

Let us expand each $W_{\alpha \beta}$ on this basis

$$\hat{W}_{\alpha \beta}(t) = \sum_{j=1}^{N^2} (\hat{F}_j, \hat{W}_{\alpha \beta})$$

Then

$$V(t) \rho_s = \sum_{j, k = 1}^{N^2} C_{jk}(t) \hat{F}_j \rho_s \hat{F}_k^+$$

$$C_{jk}(t) = \sum_{\alpha \beta} (F_j, W_{\alpha \beta}(t))(F_k, W_{\alpha \beta}(t))^*$$

The matrix $C_{jk}$ is hermitian and positive by construction.
We can now write \( Lp_s \) as:

\[
Lp_s = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (V(\varepsilon)p_s - p_s)
\]

By defining \( a_{jk} \) as \((a_{jk} \text{ is hermitian and positive})\)

\[
a_{nn} = \lim_{\varepsilon \to 0} \frac{C_{nn}(\varepsilon) - N}{\varepsilon}
\]

\[
a_{jn} = \lim_{\varepsilon \to 0} \frac{C_{jn}(\varepsilon)}{\varepsilon}
\]

\[
a_{jn} = \lim_{\varepsilon \to 0} \frac{C_{jn}(\varepsilon) - N}{\varepsilon}
\]

\[
\hat{F} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N^2} a_{jn} \hat{F}_j
\]

\[
\hat{G} = \frac{1}{2N} A_{NN} \hat{1}_s + \frac{1}{2} (\hat{F}^+ + \hat{F})
\]

\[
\hat{H} = \frac{1}{2\hat{c}} (\hat{F}^+ - \hat{F})
\]

we have

\[
Lp_s = -i [H, p_s] + \left\{ G, p_s \right\} + \sum_{j,k=1}^{N^2} a_{jk} \hat{F}_j p_s \hat{F}_k^+
\]

From \( \text{Tr}_s (Lp_s) = 0 \) \( \forall p_s \) we have

\[
G = -\frac{1}{2} \sum_{j,k=1}^{N^2} a_{jk} \hat{F}_j^+ \hat{F}_k
\]

then

\[
Lp_s = -i [H, p_s] + \sum_{j,k=1}^{N^2} a_{jk} \left( \hat{F}_j p_s \hat{F}_k^+ - \frac{1}{2} \left\{ \hat{F}_j^+ \hat{F}_k, p_s \right\} \right)
\]
Since $\hat{a}_j$ is positive and Hermitian, it can be diagonalized by a unitary transform 
\[
\hat{u} \cdot \hat{a}_j \hat{u}^+ = \begin{pmatrix} \tilde{\gamma}_j & 0 \\ 0 & \tilde{\gamma}_j^{*} \end{pmatrix}, \quad \tilde{\gamma}_j \geq 0
\]

Define then 
\[
\hat{F}_j = \sum_{k=1}^{n^2} \bar{u}_{kj} \hat{A}_k
\]

one can finally express the Liouvilleian in diagonal form as
\[
\dot{\rho}_s = -i [H, \rho_s] + \sum_{k=1}^{n^2} \gamma_k (\hat{A}_k \rho_s \hat{A}_k^+ - \frac{1}{2} \hat{A}_k^+ \hat{A}_k \rho_s - \frac{1}{2} \rho_s \hat{A}_k^+ \hat{A}_k)
\]

which is the most general form of the generator of a quantum dynamical semigroup.

$A_k$: Lindblad operators
$\gamma_k$: (inverse time) relaxation rates, linked to correlation functions of the environment dissipator: 
\[
D(\rho_s) = \sum_{k} \gamma_k (\hat{A}_k \rho_s \hat{A}_k^+ - \frac{1}{2} \hat{A}_k^+ \hat{A}_k \rho_s - \frac{1}{2} \rho_s \hat{A}_k^+ \hat{A}_k)
\]

\[
\frac{d\rho_s(t)}{dt} = -i [H, \rho_s(t)] + D(\rho_s(t))
\]
Operators $H$ and $A_n$ are not linked to Hamiltonian and local system $A_n$ in general. They are also not unique. $L$ is invariant under the transformations:

1. Unitary Transform of $A_n$
   \[ \sqrt{U} A_n \sqrt{V} = \Sigma_j M_{nj} \sqrt{V_j} A_j \]

2. Inhomogeneous Transform
   \[ A_n \rightarrow A'_n = A_n + a_n \]
   \[ H \rightarrow H' = H + \frac{1}{2i} \Sigma_j Y_j \left( a^* A_j - a \cdot A_j^* \right) + b \]
   \[ a_n \in \mathbb{C}, \quad b \in \mathbb{R} \]

Also possible to have time-dependent generator
\[ \frac{d}{dt} \rho_s(t) = L(t) \rho_s(t) \]

Then
\[ V(t, t_0) = T \exp \left( \int_{t_0}^{t} ds \ L(s) \right) \]
\[ \frac{d}{dt} V(t, t_0) = L(t) V(t, t_0) \]
\[ V(t, t_0) = V(t, t_1) V(t_1, t_0) \]
How to derive $H$ and $A$ explicitly.
Possible within given approximation schemes.

1. **Weak coupling limit**

   Interaction between $S$ and $B$ is weak so that $p_B$ is approximately not affected by the system.

   * Introduce 2nd order Born approx
     \[ p(t) = p_s(t) \otimes p_B \]

   True only on a coarse-grained time: $S$ affects $B$ but $B$ decays very rapidly back to $p_B$

   * Introduce Markov approximation
     \[ p_s(s) \text{ retarded time} = p_s(t) \]

   Together, they for the Born-Markov approx.
   Condition: both correlated. Time $T_K < T_R$
   $T_K$ rel time of the system, over which the state of $S$ varies appreciably.

   Secular or rotating-wave approx. Requires
   \[ \gamma_s \approx |\omega - \omega'|^{-1} \ll T_R \]
In this limit therefore, we need both the internal system dynamics and the bath dynamics to be much faster than the relax time $T_R$ induced by the bath onto the system.

Then:

$$\frac{d \rho_s(t)}{dt} = -i \left[ H + H_{LS}, \rho_s(t) \right] + D(\rho_s(t))$$

$$H_{LS} = \sum_{\omega} \sum_{k \beta} \Gamma_{k \beta}(\omega) A_\beta^+(\omega) A_\beta(\omega)$$

Lamb–shift Hamiltonian

$$D(\rho_s) = \sum_{\omega} \sum_{k \beta} \Gamma_{k \beta}(\omega) \left( A_\beta(\omega) \rho_s A_\beta^+(\omega) - \frac{1}{2} \left[ A_\beta^+(\omega) A_\beta(\omega) \rho_s \right] \right)$$

where $\Gamma_{k \beta}(\omega) = \frac{1}{2} \gamma_{k \beta}(\omega) + i S_{k \beta}(\omega)$

are correl. functions of the reservoir

$$\Gamma_{k \beta}(\omega) = \int_0^{\infty} ds \ e^{i \omega s} \langle B_\beta^+(s) B_\beta(0) \rangle$$

$$H_S = \sum_\alpha \hat{A}_\alpha \otimes \hat{B}_\alpha$$

$$A_\alpha(\omega) = \sum_{\omega'} \Gamma(\omega') A_\alpha \Gamma(\omega')$$

$$\omega = 3 - \omega$$
Very important. Environment induces transitions through system eigenstates $\Pi(E)$ is the projector on the eigenstate system subspace at energy $E$.

In particular

\[ [H_s, A_\alpha(w)] = -\omega A_\alpha(w) \]
\[ [H_s, A_\alpha^+(w)] = +\omega A_\alpha^+(w) \]

In most treatments of open many-body QCD, this is neglected, and local (i.e. non-interacting) operators are used instead.

Example: 1 cavity mode

\[ H_s = i \omega a^+ a \]

Then $\hat{a}$ and $\hat{a}^+$ are eigensoperators then

\[ \phi_s = -i [H_s, \phi_s] + \Gamma (a^+ a^+ \frac{1}{2} \hat{a} \hat{a}^+ - \frac{1}{2} \hat{a}^+ \hat{a} a^+ a^+ \phi_s) \]

This is correct. However two cavity modes

\[ H_s = i \omega, a^+ a^+ + i \omega \hat{a}^+ \hat{a} a^+ + \Gamma (a^+ a^+ a^+ \hat{a} a^+ + \hat{a}^+ \hat{a} a^+ a^+) \]
we will always see

\[ D(\rho_s) = \sum_{j=1}^{N} \delta_j \left( a_j \rho_s a_j^+ - \frac{1}{2} a_j^+ a_j \rho_s - \frac{1}{2} \rho_s a_j a_j^+ \right) \]

but this is not in line with what seen previously because eigen-operators are \( \tilde{A}_j = U_j \tilde{a}_j U_j^+ \) (because of the coupling)

Ideally, solving a model of an open quantum system requires to diagonalize the system Hamiltonian first.
Example: Quantum optical
matter eq.

\[ H_B = \sum_k \sum_{\alpha=1/2} t_{\alpha\alpha} \hat{a}_\alpha^+(k) \hat{a}_\alpha(k) \]

Electrom. field

\[ \hat{E} = i \sum_k \sum_{\alpha=1/2} \frac{2\omega_k \varepsilon_k}{V} \left( \varepsilon_0^0(k) \hat{a}_\alpha^+(k) - \hat{a}_\alpha(k) \right) \]

\[ \hat{H}_E = -\vec{D} \cdot \hat{E}, \quad \vec{D} \text{ dipole q. of the system} \]

\[ \hat{A}(\omega) = \sum_{\varepsilon' < \varepsilon} GG(\varepsilon) \hat{D}(\varepsilon') \]

\[ \hat{D}(\varepsilon) = \sum_\omega e^{-i\omega t} \hat{A}(\omega) \]

Assuming a thermal bath

\[ \rho_B = \frac{1}{Z_0} e^{-\beta H_B}, \quad \beta = \frac{1}{k_B T} \]

\[ \langle \hat{b}_\alpha^+(k) \hat{b}_\alpha(k') \rangle = \delta_{k,k'} \delta_{\alpha\alpha} N(\omega_k) \]

\[ \langle \hat{b}_\alpha(k) \hat{b}_\alpha^+(k') \rangle = \delta_{k,k'} \delta_{\alpha\alpha} 1 + N(\omega_k) \]

\[ N(\omega) = \frac{1}{e^{\beta \omega} - 1} \]
one finally gets

\[ \frac{dρ_ς}{dt} = -\frac{i}{\hbar} \left[ H + H_{\text{ws}}, ρ_ς \right] + D(ρ_ς) \]

\[ H_{\text{ws}} = \sum_ω t_ω S(ω) \bar{A}^+(ω) \cdot \bar{A}(ω) \]

\[ D(ρ_ς) = \sum_ω \frac{4ω_0^3}{3τc^3} (1 + N(ω)) \left( \bar{A}(ω)ρ_ς \bar{A}^+(ω) - \frac{1}{2} \left\{ \bar{A}(ω) \bar{A}^+(ω), ρ_ς \right\} \right) \]

\[ + \sum_ω \frac{4ω_0^3}{3τc^3} N(ω) \left( \bar{A}^+(ω)ρ_ς \bar{A}(ω) - \frac{1}{2} \left\{ \bar{A}(ω) \bar{A}^+(ω), ρ_ς \right\} \right) \]

\[ S(ω) = \text{structure factor} \]

\[ = \frac{2}{3π\hbar c^3} \rho \int_0^{100} dω_0 \frac{ω_0^3}{ω - ω_0} \left( \frac{1 + N(ω_0)}{ω + ω_0} + \frac{N(ω_0)}{ω + ω_0} \right) \]

\[ S(ω) \text{ results from the density of both modes. For a flat density one would have } S(ω) = 0 \]

Examples: phonons, both phonon bath.
Exercise

Numerically solve

\[
\frac{d\rho}{dt} = -i \left[ H, \rho \right] + \kappa \sum_j \left( a_j \rho a_j^\dagger - \frac{1}{2} a_j^\dagger a_j \rho - \frac{1}{2} \rho a_j^\dagger a_j \right)
\]

\[
H = \hbar \omega \sum_j a_j^\dagger a_j + \sum_j (a_j a_j^\dagger + a_j^\dagger a_j)
\]

\[
+ U \sum_j a_j^2 a_j^\dagger + \sum_j (F e^{-i\omega t} a_j^\dagger + F^* e^{i\omega t} a_j)
\]

Set \( a_j^\dagger \rightarrow a_j e^{-i\omega t} \quad a_j \rightarrow a_j e^{i\omega t} \)

\[
H = -\Delta \sum_j a_j^\dagger a_j - \sum_j \omega_e \sum_j (a_j^\dagger + a_j)
\]

\( \Delta, \omega_e, T, F \in \mathbb{R} \)

Define quantities w.r.t. \( \kappa \) (i.e., set \( \kappa = 1 \))

Vectorization \( \rho \rightarrow |\rho\rangle \)

\[
|xy\rangle \rightarrow Y^{T} \otimes |x\rangle \langle y| \quad |x\rangle \langle y| \rightarrow Y \otimes |x\rangle \langle y|
\]

\[
L = -i \left( \mathbb{I} \otimes H - H^T \otimes \mathbb{I} \right) + \sum_j \left( a_j^\dagger \otimes a_j - \frac{1}{2} \mathbb{I} \otimes a_j a_j^\dagger - a_j^\dagger a_j \otimes \mathbb{I} \right)
\]
For one boson,
\[ a^+ = \sum_{n=0}^{N_{\text{max}}-1} \sqrt{n+1} | n+1 \rangle \langle n | = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \sqrt{2} & 0 & \cdots & 0 \\
0 & 0 & 0 & \sqrt{3} & \cdots & 0 \\
0 & 0 & 0 & 0 & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ddots & \ddots \\
\end{pmatrix} \]
Stochastic dynamics in Hilbert space

$P_s(t)$ is a statistical description of the state of the system.

Can we define a stochastic process for $|\Psi(t)\rangle$ such that

$$P_s(t) = \mathbb{E}(|\Psi(t)\rangle\langle\Psi(t)|)$$

"Unravelling of the master equation" 

Physical meaning: continuous measurement theory: stochastic dynamics produced by continuously monitoring a certain observable of the environment.

The stochastic process is then defined by a piecewise deterministic process

Closed system

$$U(t, t_0) = e^{-i\mathcal{H}(t-t_0)}$$

$$P[\Psi(t)] = P_0[U^{-1}(t,t_0)\Psi]$$

$P_0[\cdot]$ initial PDF

If $P_0 = \delta(x) \Rightarrow |\Psi(t)\rangle$ is fully deterministic.
A fully deterministic state process is per def Markov.

In terms of functional int.

\[ P[\psi, t] = \int D\psi_0 D\psi_0^* \rho_0[\psi_0] \delta[\psi - \mathcal{U}(\psi_0, t_0)] \]

\[ \rho_0[\mathcal{U}(\psi_0, t_0)] \]

Differential form

\[ \frac{\delta}{\delta \psi(x)} \psi(y) = \frac{\delta}{\delta \psi^*(x)} \psi^*(y) = \delta(x-y) \]

\[ \sum \frac{\delta}{\delta \psi(x)} \psi^*(x) = 0 \]

\[ \frac{d}{dt} F[\psi(t)] = \int dx \left( \frac{\delta F}{\delta \psi(x)} \frac{d\psi(x,t)}{dt} + \frac{\delta F}{\delta \psi^*(x)} \frac{d\psi^*(x,t)}{dt} \right) \]

\[ \frac{\partial}{\partial t} P[\psi, t] = i \int dx \left( \sum \frac{\delta}{\delta \psi(x)} (H \psi(x) - \frac{\delta}{\delta \psi^*(x)} (H \psi^*(x)) \right) \times P[\psi, t] \]

Liouville eq. corresp. to

\[ \psi(t) = -iH \psi(t) \]
Quantum master equation

\[ \frac{d}{dt} p_s(t) = -i \left[ H, p_s \right] + \sum_i \delta_i \left( A_i p_s A_i - \frac{1}{2} A_i^+ A_i p_s - \frac{1}{2} p_s A_i^+ A_i \right) \]

\[ p_s(\psi, t) = \int D\bar{\psi} D\psi \ast T[\psi, t/\psi, t_0] p[\psi, t_0] \]

\[ p_s(x, \psi^*, t) = E[\psi(x, t) \psi^*(x, t)] = \int D\psi D\psi^* p(\psi, t_0 \psi^*, t_0) \]

We show that this Liouville eq. coincides with the QME

\[ \frac{\partial}{\partial t} p[\psi, t] = i \int d\bar{x} \left( \delta_{\bar{x}} G(\psi)(\bar{x}) - \delta_{\bar{x}}^* G(\psi)^*(\bar{x}) \right) p[\psi, t] \]

\[ + \int D\tilde{\psi} D\tilde{\psi}^* \left( W(\psi/\tilde{\psi}) p[\tilde{\psi}, t] - W(\psi^*/\tilde{\psi}) p[\tilde{\psi}, t] \right) \]

1st term:

\[ \frac{d}{dt} \psi(t) = -i G(\psi(t)) \]

\[ = -i \tilde{H} \psi(t) + \frac{1}{2} \sum_i \delta_i \left( A_i^+ \psi(t) \psi(t)^* A_i \right) \]

\[ \tilde{H} = H - \frac{i}{2} \sum_j \delta_j A_j^+ A_j \]
It is a non-hermitian law, implying a decay. This 
The nonlinear term in $G(\Psi)$ is then 

to preserve the norm of $\Psi$. 
The eq. $\Psi = -i G(\Psi)$ is then norm 

preserving. 

In the Liouville eq. there are gain 
and loss terms. The transfer rate is 
defined as 

$$ W(\Psi|\Psi') = \sum_j \frac{\gamma_j}{\|A_i\|} \|\varphi_j\|^2 s \left( \frac{A_i \Psi_j}{\|A_i\| \varphi_j} - \Psi_j \right) $$

which defines jumps from $\Psi$ to 
$\frac{A_i \Psi_j}{\|A_i\| \varphi_j}$ with rates $\frac{\gamma_j}{\|A_i\|} \|\varphi_j\|^2$

Total transfer rate from $\Psi$ : $\Gamma(\Psi)$

$$ \Gamma(\Psi) = \int D\Psi D\Psi' W(\Psi|\Psi') = \sum_j \frac{\gamma_j}{\|A_i\|} \|\varphi_j\|^2 $$
The formal solution between two jumps is
\[ \Psi(x, t) = \frac{e^{-iHt} \Psi}{\| e^{-iHt} \Psi \|} \]

Deriving, one obtains \( \Psi \rightarrow iG'(t) \)

So one can solve the linear Schröd. eq. and take the norm at each time step.

To show that one obtains again the QME
\[ \frac{\partial}{\partial t} \rho_s(x, t) = \int d^4 y \delta(x - y) \nabla^2 \rho_s(y) \]
\[ = \frac{\partial}{\partial t} \rho_s(x, t) + \frac{\partial}{\partial t} \rho_s^*(x, t) \]
\[ = \int d^4 y \delta(x - y) \nabla^2 \rho_s(y) \]
\[ = \int d^4 y \delta(x - y) \nabla^2 \rho_s(y) \]
\[ = -i \int d^4 y \delta(x - y) \nabla \cdot \nabla \rho_s(y) \]
\[ = -i \int \delta(x - y) \left[ i \left( \nabla \cdot \nabla \rho_s(y) \right) - \frac{1}{2} \nabla^2 \rho_s^*(y) \right] d^4 y \]
\[ = \sum_j \left( \frac{\partial}{\partial t} \rho_{s_j}^* \right) + \sum_j \partial_j E \left[ \| A_j \psi \|^2 \psi^*(x) \psi(x) \right] \]
\[ + \sum_j \partial_j E \left[ \| A_j \psi \|^2 \psi^*(x) \psi(x) \right] \]
\[ \frac{\partial}{\partial t} \rho_{s_j} = \sum_j A_j \rho_{s_j} A_j^* - \sum_i \psi_i E [ \| A_j \psi \|^2 \psi^*(x) \psi(x) ] \]
The mean
\[ \frac{d}{dt} \rho_s = -i \left[ \hat{H}, \rho_s \right] + \sum_j \eta_j A_j \rho_s A_j^+ \]

Waiting time distrn.

Total rate:
\[ \Gamma(\Psi(t)) = \int d\Psi^* \psi W(\Psi(\Psi(t)) = \sum_j \frac{\eta_j}{\|A_j \Psi(t)\|^2} \]

Suppose \( \Psi \) was reached with a jump at time t. At time \( t + \delta t \), rate of jump:
\[ \Gamma(\Psi(t)) = \sum_j \frac{\eta_j}{\|A_j \Psi(t)\|^2} \]

In fact:
\[ \frac{d}{dt} \| e^{-\hat{H} t} \psi \|^2 = i \frac{d}{dt} e^{\hat{H} t} \psi \left( \hat{A} \psi + \hat{A}^+ \psi \right) \]

\[ = -\sum_j \frac{\eta_j}{\|A_j \psi\|^2} \left( \frac{\psi |A_j \psi|^2}{\|A_j \psi\|^2} \right) \]

but \[ e^{\hat{H} t} \psi = \Psi(t) \] Then
\[ \frac{d}{dt} \| e^{-\hat{H} t} \psi \|^2 = \sum_j \frac{\eta_j}{\|A_j \psi\|^2} \left( \frac{\psi |A_j \psi|^2}{\|A_j \psi\|^2} \right) \]
$W(\psi|\tilde{\psi})$ is the probability current from $\tilde{\psi}$ to $\psi$. If before the jump the state is $\tilde{\psi}$, then there are a discrete set of possible jumps.

\[
\tilde{\psi} \rightarrow \psi = \frac{A_j \tilde{\psi}}{||A_j \tilde{\psi}||} \quad \mathcal{P}_j = \frac{\gamma_j ||A_j \psi||^2}{\Gamma(\tilde{\psi})}
\]

There are the "quantum jumps".

We can translate the Liouville eq. into a stochastic diff. eq.

\[
d\psi(t) = -i \mathcal{L}(\psi(t)) + \sum_j \left( \frac{A_j \psi(t)}{||A_j \psi(t)||} - \psi(t) \right) dN_j(t)
\]

$dN_j(t)$ stochastic Poisson increments

\[
dN_j(t) dN_k(t) = \delta_{jk} dt \quad dN_j(t)
\]

\[
E[dn_j(t)] = \gamma_j ||A_j \psi(t)||^2 dt
\]

the process $N_j(t)$ counts the number of jumps of type $j$. Its diff. $dN_j(t)$ defines the stochastic process of the jumps of type $j$.
\( \rho(t) \) is a statistical description of the state of the system.

Can we define a stochastic process in Hilbert space \( |\psi(t)\rangle \) such that

\[
\rho(t) = \mathbb{E}\left( |\psi(t)\rangle \langle \psi(t)| \right)
\]

This is possible in several different ways and it's called "unravelling of the master eq."

The physical meaning is that the stochastic dynamics is the result of continuous measurements of a certain observable of the environment. Each time the environment is measured, the full state of system + environment collapses into a tensor product state \( |\psi(t)\rangle \otimes |\psi_E(t)\rangle \) which makes it possible to define \( |\psi(t)\rangle \) at each \( t \). The result is a (piecewise) deterministic process for \( |\psi(t)\rangle \), called quantum trajectory.
Several books and articles deal with this. E.g.

- Breuer, Petruccione: "The theory of open quantum systems"
- Gardiner, Zoller: "Quantum Noise; ..."
- Plefka, Knight: RMP 70, 101 (1998)

Notice that the simplest QSDDE is the Schrödinger equation:

$$\frac{d}{dt}|\psi(t)\rangle = -i \hat{H} |\psi(t)\rangle$$

This is non-stochastic. The corresponding master eq. is the von Neumann eq.

$$\frac{d}{dt}\rho(t) = -i [\hat{H}, \rho(t)]$$

This, if interpreted as a stochastic process, is a Markovian process, because the evolution from time t is fully deterministic and thus only depends on the condition $\rho(t)$ and not on the past history.
Instead of deriving a general formal theory of QSDE, we will derive the ME from the QSDE in a very simple case, as an example. Then we will discuss all other cases briefly.

Consider a single bosonic mode subject to damping. The ME is

$$\frac{d\rho}{dt} = -i[H, \rho] - \frac{1}{2}(a^+a \rho + \rho a a^+ - 2a \rho a^+)$$

We show that the stochastic diffusion equation

$$d|\Psi(t)\rangle = \left[ dN(t) \left( \frac{a}{\langle \Psi(t) | a \Psi(t) \rangle} - 1 \right) + dt \left( \frac{\langle \Psi(t) | a^+ a \Psi(t) \rangle}{2} - iH \right) \right] |\Psi(t)\rangle$$

is equivalent to the ME with $p(t) = \langle \Psi(t) | \Psi(t) \rangle$ provided $dN(t)$ is a random Poissonian number process, characterized by:

$$\mathbb{E}[dN(t)] = \langle n(t) \rangle dt = \langle \Psi(t) | a \rho a^+ | \Psi(t) \rangle$$

$$\langle dN(t) \rangle^2 = dN(t) \quad \text{(random variable taking only values 0, 1)}$$

Before we proceed, notice two things. First, the QSDE is norm preserving, i.e., $\langle \Psi(t) | \Psi(t) \rangle = 1$ for
In general, if we have an unnormalized QSDE
\[ d |\Psi\rangle = \left[ A dt + (B-1) dN \right] |\Psi\rangle \]
The corresponding normalized QSDE is
\[ d |\Psi\rangle = \left[ \left( A - \frac{1}{2} \langle A+A^+ \rangle \right) dt + \left( \frac{B}{1+N} - 1 \right) dN \right] |\Psi\rangle \]
We could use the unnormalized QSDE for our proof, but we would need to define
\[ \mathcal{P}(t) = \mathcal{E} \left[ \frac{|\Psi(t)\rangle \langle \Psi(t)|}{\langle \Psi(t)|\Psi(t)\rangle} \right] \]
instead.

The second remark is that the QSDE here is written in Itô form. We don't make
here an intro to stochastic calculus. The
main difference to ordinary calculus is
the dependence of the increment on the
integrand. In Itô:
\[ \int_{0}^{t} g(t') dN(t') = \lim_{n \to \infty} \sum_{i=0}^{n} g(t_j) \left[ N(t_{i+1}, 0) - N(t_i, 0) \right] \]
with \( 0 < t_0 < \ldots < t_j \)
First, for quantum variables, \( [g(t'), dN(t')] = 0 \)
because the increment is at \( t \geq t_j \) and
is therefore uncorrelated to \( dN(t_j) \) which only
depends on \( t < t_j \) (Markov)
However, for Itô's, if $a$ and $b$ are stock variables, then

$$d(ab) = ab \, dt + ab \, da + da \, db$$

In fact, an SDE has the form

$$da = A \, dt + (B-1) \, dN$$

with $dN^2 \propto dt$ thus $dN = \sqrt{dt}$

If we want 1st order in $dt$ of $d(ab)$, then we must include the $dN^2$ contribution from $da \, db$.

So rules for Itô's calculus are different from ordinary calculus.

To avoid this there would be Stratanovich stochastic calculus, where the increment is defined as:

$$\int_{t_0}^{t_1} \xi(t') \, dN(t') = \lim_{n \to \infty} \left[ \sum g(\bar{t}_i) \left[ B(\bar{t}_{i+1}) - B(\bar{t}_i) \right] + \sum g(\bar{t}_i) \left[ B(t_i) - B(t_{i-1}) \right] \right]$$

where $\bar{t}_i = \frac{1}{2} (t_i + t_{i+1})$

In this case the increment is $O(dt)$ and $d(ab) = ab \, dt + ab \, da$

but $[g(t'), dN(t')] \to 0$.

We stick to Itô's
From the QSDE one has:  
\[ d|\Psi (t)\rangle = (d|\Psi (t)\rangle) \langle \Psi (t) + 1 | \Psi (t) + 1 \rangle \]  
\[ + (d|\Psi (t)\rangle) \langle \Psi (t) + 1 | \Psi (t) + 1 \rangle (d|\Psi (t)\rangle) \langle \Psi (t) + 1 | \Psi (t) + 1 \rangle \]  
\[ = \left[ \frac{dN}{(\sqrt{2\gamma})} - 1 \right] + \frac{dt}{(\sqrt{2\gamma})} \left( \frac{\langle \Psi (t) \rangle}{(\sqrt{2\gamma})} - \frac{u}{(\sqrt{2\gamma})} + iH \right) \right] \]  
\[ + \frac{dN}{(\sqrt{2\gamma})} \left( \frac{\langle \Psi (t) \rangle}{(\sqrt{2\gamma})} - 1 \right) \]  
\[ + \frac{dN}{(\sqrt{2\gamma})} \left( \frac{\langle \Psi (t) \rangle}{(\sqrt{2\gamma})} - 1 \right) \]  

Use  
\[ dN(t)^2 = dN(t) \]  
\[ n = a^\dagger a \]  
\[ dp_c(t) = dN(t) \left( \frac{a^\dagger \rho_c a}{\langle \Psi (t) \rangle} - \rho_c \right) + \frac{dt}{(\sqrt{2\gamma})} \left( \frac{\langle \Psi (t) \rangle}{(\sqrt{2\gamma})} - \frac{u}{(\sqrt{2\gamma})} - n - iH(t) \right) \]  
This is called the stochastic master equation.  
and is still another stochastic resp. of a Markovian  
QNS dynamics. Its property is to  
conservate purity, so if  
\[ \rho_c(0) = |\Psi (0)\rangle \langle \Psi (0) | \]  
then  
\[ \rho_c(t) \]  
is always a pure state  
We now take the exp. value over the stat.  
ensemble.  
Note that  
\[ E \left[ dN(t) \rho_c(t) \right] = E \left[ dN(t) \right] E \left( \rho_c(t) \right) \]  
because of Markovianity,  \[ \rho_c(t) \]  depends only  
on \[ dN(t) \]  with  
\[ t' < t. \]
Then
\[
\frac{dp(t)}{dt} = E\left[ \frac{dp_e(t)}{dt} \right] = -i \left[ H_{eff}, p \right] dt + \alpha p a^+ dt
\]
\[
\Rightarrow \frac{dp}{dt} = -i \left[ H_{eff}, p \right] + \alpha p a^+
\]

\[H_{eff} = \mathcal{H} - \frac{\alpha}{2} a^+ a\]

which is the ME.

We can generalize to a stochastic rate \( \gamma \) by defining \( E[dN(t)] = \gamma \langle n \rangle dt \)

The process \( dN(t) \) can be interpreted as a jump process, namely

\[
dN(t) = \begin{cases} 1 & \text{if one boson is lost in } [t, t+dt] \\ 0 & \text{if not} \end{cases}
\]

The generalization to several modes and decay channels is:

\[
d[\psi(t)] = \left( -i H_{eff} \psi(t) + \frac{1}{2} \sum_j \gamma_j \langle \psi | A_j^+ A_j | \psi \rangle \psi(t) \right) dt
\]

\[
+ \sum_j \left( \frac{A_j^+}{\langle \psi | A_j^+ A_j | \psi \rangle} - 1 \right) dN_j(t) \langle \psi | \psi \rangle dt
\]

\[
dN_j(t) dN_k(t) = \delta_{jk} dN_j(t)
\]

\[E[dN_j(t)] = \gamma_j \langle \psi_j(t) | A_j^+ A_j | \psi_j(t) \rangle dt\]
\[ H_{eff} = H - \frac{i}{\hbar} \sum_j A_j A_j^\dagger \]

The interpretation in terms of quantum measurement here is that of "photodetection." We count quanta emitted into the environment through the transition ("quantum jump") generated by $A_j$.

The total jump rate for the non-normalized $|\Phi(t)\rangle$ is

\[ \Gamma (|\Phi(t)\rangle) = \sum_j \gamma_j \langle \Phi | A_j A_j^\dagger | \Phi \rangle \]

From this one can show that, if a jump occurred at time $t$, the rate of the next jump occurring at $t + \tau$ is

\[ \Gamma (|\Phi(t)\rangle) = -\frac{d}{d\tau} \ln \| e^{-i\int_0^\tau H_{eff} dt} |\Phi(t)\rangle \|^2 \]
From \( P(\Psi(x)) \) one can derive the distr.
function of the random waiting time \( \tau \):
\[
F(\bar{\Psi}, \tau) = 1 - \exp \left( -\int_0^\tau ds \frac{1}{\Psi(s)} \right)
\]
\[
= 1 - \frac{\| \exp(-i \tilde{A} \tau) \bar{\Psi} \|^2}{\| \bar{\Psi} \|^2}
\]
This is the prob. that the next jump occurs between \( t \) and \( t+\tau \).

The stock process \( \tilde{P}(\bar{\Psi}, t) \) can therefore be simulated as follows:

1. Assume that the normalized state \( \bar{\Psi}^* \) was reached at \( t \).
2. Determine a random waiting time according to \( F(\bar{\Psi}, \tau) \). Eg. \( \eta = \text{rand}(0, 1) \) determine \( \tau \) from
\[
\eta = 1 - F(\bar{\Psi}, \tau) = \frac{\| e^{-i \tilde{A} \tau} \bar{\Psi} \|^2}{\| \bar{\Psi} \|^2}
\]
within \( t \leq t+\tau \),
\[
\Psi(t+s) = \frac{e^{-i \tilde{A} s} \bar{\Psi}}{\| e^{-i \tilde{A} s} \bar{\Psi} \|}, \quad 0 \leq s \leq \tau
\]
3. At time \( t+\tau \) draw randomly one jump according to
\[
\rho_j = \frac{\delta_j \|A_j \psi(t+r)\|^2}{\sum_j \delta_j \|A_j \cdot \psi(t+r)\|^2}
\]

and replace

\[\psi(t+r) \rightarrow \frac{A_j \cdot \psi(t+r)}{\|A_j \cdot \psi(t+r)\|}\]

4. Repeat 1 to 3

5. Average over several trajectories \(\psi^r(t)\), \(r = 1, 2, \ldots, R\) obtained in this way.

Comments on Determining the Waiting Time
Diffusion limit of the Stock Sch. eq. \[ A = I + \varepsilon C \quad \varepsilon \to 0 \]

Time limit of terms arbitrarily large \( \varepsilon \to 0 \)

It can be shown that in this form, one gets

\[ d\Psi(t) = -iK(\Psi(t))dt + \sqrt{\gamma_0} \Lambda(\Psi(t))dW(t) \]

(in Itô form)

\( dW(t) \) is the increment of a real Wiener process.

\[ K(\Psi) = H\Psi + i\bar{\delta}(\overline{<C>}) \left(C - \frac{1}{2} <C>_t^2 - \frac{1}{2} C^2\right) \Psi \]

\[ \Lambda(\Psi) = \left(C - <C>_t\right) \Psi \]

\[ \gamma_0 = \varepsilon^2 \gamma_0 \text{ rate} \]

\[ dW_i(t) dW_j(t') = \delta_{ij} dt \]

Then \( \langle (SW)^2 \rangle \approx \Delta t \)

and \( SW \propto \sqrt{\Delta t} \)

Comments on efficiency of expr. mean.
The stock dynamics that unravels the QSE is not unique. E.g., $H, A_j, \chi_j \rightarrow H', \chi_j', A_j'$ will give a different stock approach.

Interpretation as cont. measurement:
1. Photodetection $\rightarrow$ quantum jumps
2. Homodyne photodetection $\rightarrow$ stock Schrödinger
3. Heterodyne photodetection $\rightarrow$ with a Wiener process.
Numerical methods for QQS: overview

We may need to solve one of the problems:

1. Dynamics: \[ \dot{\rho} = L\rho \]
2. NESS: \[ L\rho = 0 \text{ with } \text{Tr}(\rho) = 1 \]

Remarks:

- Liouville super-operator \( L \) is non-hermitian has complex eigenvalues. If \( \lambda \) is ev, \( \lambda^* \) is.
  Some ev are real only
  \[ \text{Re}(\lambda) \leq 0 \text{ dissipative dynamics} \]
  \[ \text{Re}(\lambda) = 0 \text{ NESS} \]

Existence and/or uniqueness of NESS is not obvious or granted!

D. Nigro, arXiv: 1803.06279

F. Minganti et al. arXiv: 1804.11293

- To numerically solve, it is almost always useful to adopt Choi isomorphism, aka vectorization. Map operators on \( \mathcal{H} \) to vectors on \( \mathcal{H} \otimes \mathcal{H} \)

\[ \rho = \sum_{i\in\mathcal{M}} \rho_{ii} |i\rangle\langle i| \rightarrow |\rho\rangle = \sum_{i\in\mathcal{M}} \rho_{ii} |i\rangle \otimes |i\rangle \]
i.e. concatenate the columns of $p$

For vectorization, given operators $X, Y$,

$|X \rho Y \rangle \rightarrow (Y^\top \otimes X |\rho \rangle$

Then the Liouvillian is mapped onto the $q$

$L = -i(I \otimes H - H^T \otimes I) + \sum_j \left[ A_j^* \otimes A_j^\dagger - \frac{1}{2} I \otimes A_j^* A_j - A_j A_j^\top \otimes I \right]$

which can be built directly into computer code.


1) We have already seen QSDFE
   Pro: $O(N)$ instead of $O(N^2)$
   Con: May not sample efficiently
       Difficulty in solving SDE

- Exact method.
  Choose a finite-dim Hilbert space
  (for Bosons, truncate to $\text{max}/\text{site}$)
  Solve $\dot{\rho} = L \rho$ as a (nonlinear) ODE.
  Pro: Direct solution, exact up to truncation
  Con: $O(N^2)$ memory limitations.
- MPO/TEBD

Based on the idea of decomposing $|\psi\rangle$ as an MPS or $|\rho\rangle$ as an MPO and applying Trotter decomposition to $L$.

Pro: Very efficient for small bond dimension (which is often the case in dissipative systems)

\[ O(N) \rightarrow O(D^4 \times d^2 \times \text{steps}) \]

Con: Inefficient if transient is highly quantum correlated

- You must be well motivated as it is hard to code

  https://www.tensors.net

- Only 1D. 2D is a mess

- Approximate methods

  - Cluster mean-field
  - Stochastic Guttmiller
  - Phase-space Methods (TWA)

Pro: Fast, not memory demanding

Con: Approx is always uncontrolled
\[ L \rho = 0 \quad \text{Tr}(\rho) = 1 \]

For finite-dim Hilbert space usually 1, only \( L = 0 \) NEST.

- **Exact:**
  Solve \( L \rho = 0 \) with concol. \( \text{Tr}(\rho) = 1 \)
  Lagrange multipliers
  More easily, replace 1st line of \( L \) with condition \( \text{Tr}(\rho) = 1 \) (in a system)

**Pro:** Exact accurate

**Con:** \( O(N^2) \)

Sometimes the system is difficult as \( L \) is bad conditioned.

Use convergence methods, eig, bieg, etc. Never good to use eigs.

- Corner space renormalization

  **Idea:**

  \[
  \begin{align*}
  \rho_1 & \xrightarrow{S} \rho_2 \\
  \rho_1 & = \sum_i \rho_{1,i} |\phi_{1,i}\rangle \langle \phi_{1,i}| \\
  \rho_2 & = \sum_i \rho_{2,i} |\phi_{2,i}\rangle \langle \phi_{2,i}| \\
  \text{Take } M \text{ highest values among } |\sum \rho_{1,i,j,k,l}| \rho_{2,j,k,l}\rangle \langle \rho_{2,j,k,l}| \end{align*}
  \]
Write all operators on the "basis"
\[ \{ | \phi_{ij} \rangle \otimes | \phi_{kl} \rangle, \quad i,j \text{ in the corner space} \}\]
Solve \( \lambda \rho = 0 \) for system \( 1 + \epsilon \)
on the corner space
Iterate.

**Pro:** Can be super efficient with the very small \( M \).

**Con:** Careful coding.
Works only for weakly entropic systems, as \( M \) grows with
\[ S = \text{Tr} \left( \rho \lambda \rho \right) \]

- **Variational QMC with neural network**
  - Nagy, VS, arXiv:1902.09483
  - Hartmann, Carleo arXiv:1902.05131
  - Vicentini et al. arXiv:1902.10104

Variational principle \[ \| \lambda \rho \|_2 \]
or \[ \| | \rho | \|_2 \]

Real-time QMC evolution with SR
Graph: No, consider formation phenomen.

Proof: O (poly (n, ε)) role step.

Solve using matrix factorization.

Let r = f (ε) with condi.

\[ \begin{align*}
(\gamma, \gamma) & = \frac{1}{k} x \cosh (\gamma \cosh (a + c x)) \\
\end{align*} \]
Variational MPO

Nasreldin Samawi, arXiv: 1504.06127
Cui et al., arXiv: 1501.06786

again represent $\rho$ as MPO

Variational sol. with DMRG

Pro: $O(\text{poly}(\text{Units}))$, very efficient if pair entanglement

Con: 1D, non-positive.