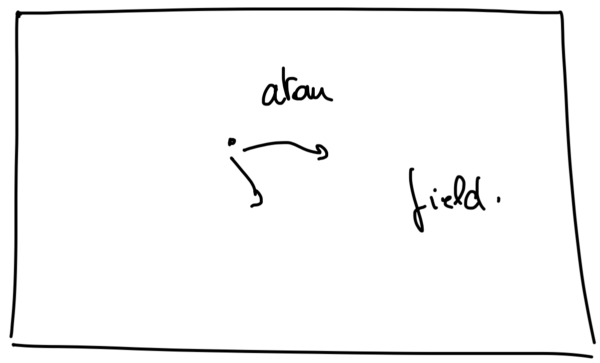
# Introduction to cavity QED.

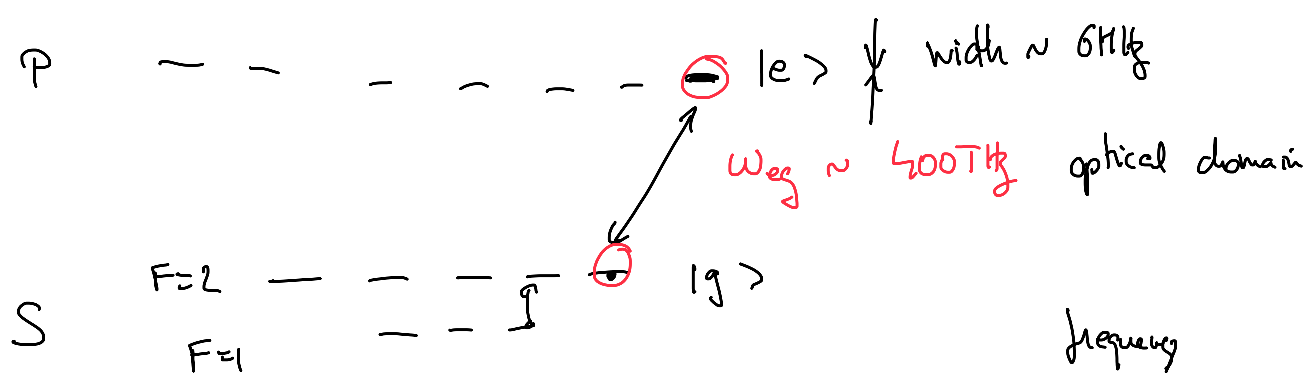
## I. Introduction

idea :

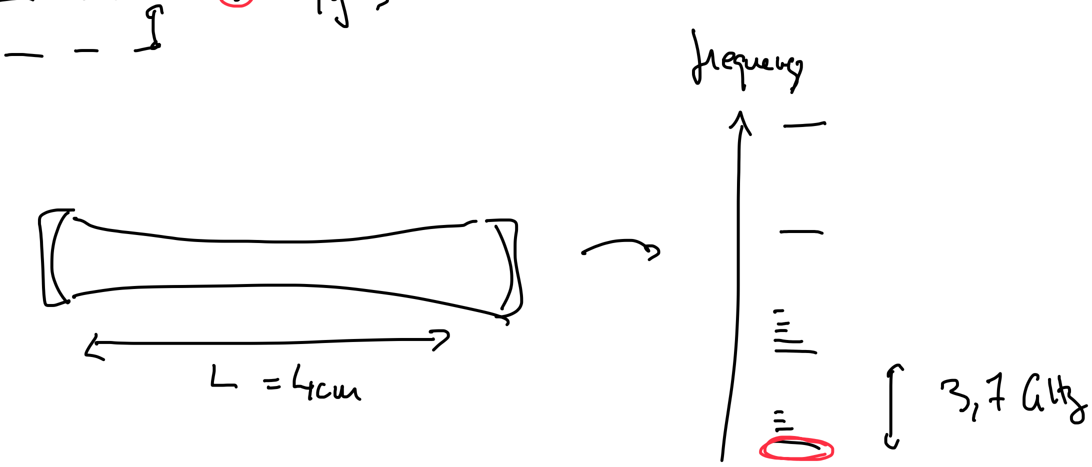



atoms: → two level systems out of the level structure of atoms

ex: $^{87}$Rb




field : cavity :




Plan: (i) Classical problem → cooperativity

(ii) Quantum case

(iii) Collective effects

# II. Classical cQED

## 1) Classical atom

elastically bound $e^-$

exposed to $\vec{E} = \frac{1}{2}\vec{e}\left(e^{i\omega t} + e^{-i\omega t}\right)E$

$\omega = c \cdot k$

$\vec{p} = \alpha \vec{E}$ $\qquad \alpha$: polarizability $\alpha(\omega)$

$$\alpha(\omega) = 6\pi\varepsilon_0 c^3 \frac{\Gamma/\omega_0^2}{\omega_0^2 - \omega^2 - i\Gamma\left(\frac{\omega}{\omega_0}\right)^3}$$

$$\Gamma = \frac{e^2 k^2}{6\pi\varepsilon_0 mc}$$ classical damping rate

$$\left( \Gamma = \frac{k^3 |d|^2}{3\pi\varepsilon_0\hbar} \qquad d = \langle e|\hat{d}|g\rangle \right)$$

Validity: · linear response $\leftrightarrow$ no saturation

$\Updownarrow$

harmonic oscillator $\qquad\qquad$ real atom

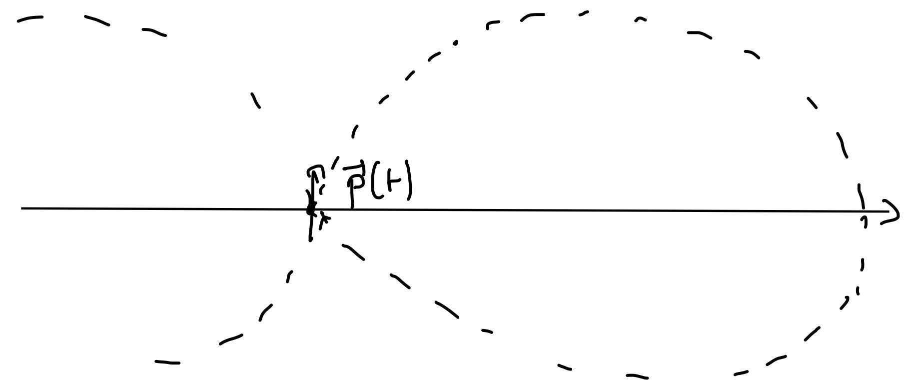


Physically:

$$|\psi(t)\rangle = |\odot\rangle + e^{i\omega_0 t}\,\varepsilon\,|\ominus\oplus\rangle$$

Properties of $\alpha$: $\quad |\alpha|^2 = \frac{6\pi\varepsilon_0}{k^3}\,\mathrm{Im}(\alpha)$

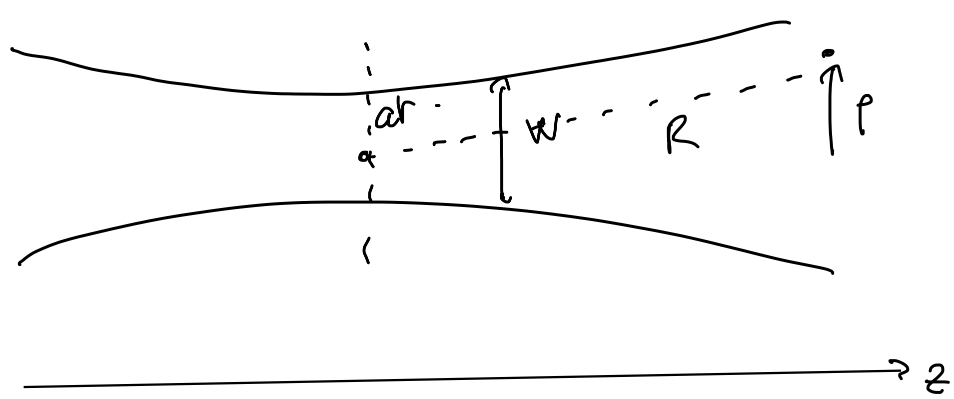
$$E(R,\theta) = \frac{k^2\cos\theta}{4\pi\varepsilon_0 R}\, e^{ikR} \cdot \alpha \cdot E$$




$(R \to \infty)$

$\theta \sim \pi/2$

## 2) Scattering of light




Gaussian beam

$$e_M(\rho,z) = \sqrt{\frac{2}{\pi w^2}}\; e^{\left(-\frac{\rho^2}{w^2} + ikz + ik\frac{\rho^2}{2z} - i\pi/2\right)}$$

Gouy ph

wave front curvature

$z \to \infty$

$$w(z) = w_0\sqrt{1+\frac{z^2}{z_R^2}}$$

Electric field: $\quad E_M = e_M(\rho,z)\cdot\frac{\mathcal{E}_M}{\sqrt{\varepsilon_0 c}}$

$$\text{Power} = \frac{|\mathcal{E}_M|^2}{2}$$

Light emitted by the atom in mode M

$$\mathcal{E}_M = \sqrt{\varepsilon_0 c}\int e_M^*\, 2\pi\rho\, d\rho\, E_{rad}$$

$$R = \sqrt{z^2+\rho^2} \sim z\left(1+\frac{\rho^2}{2z^2}\right)$$

$$\mathcal{E}_M = i\beta\,\mathcal{E} \qquad \beta = \frac{k}{\pi w^2}\,\frac{\alpha}{\varepsilon_0} \in \mathbb{C}$$

$$|\beta|^2 = \frac{6}{k^2 w^2} \operatorname{Im}\beta = \eta_{fs} \operatorname{Im}(\beta)$$

$\uparrow$ cooperativity in free space

3) Absorption, dephasing:

total radiated power: $$P_{4\pi} = \int_{S_R} dS \cdot \frac{\varepsilon c}{2} |E_{rad}|^2$$

$$= \frac{ck^4}{12\pi\varepsilon_0} |\alpha E|^2$$

$$= \operatorname{Im}(\beta) |\mathcal{E}|^2$$

$$= \frac{1}{\eta_{fs}} |\mathcal{E}_H|^2$$

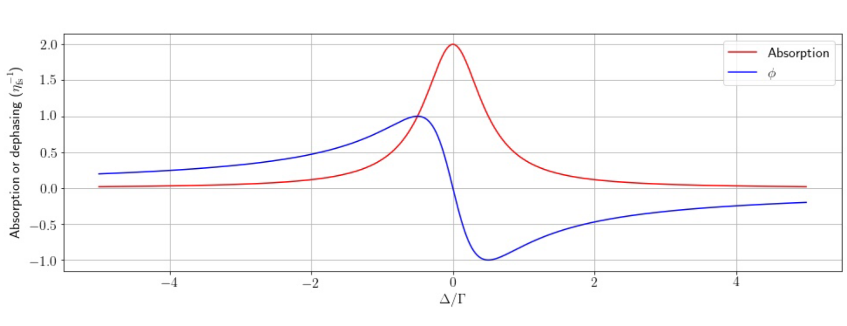
$$\boxed{\eta_{fs} = \frac{2P_H}{P_{4\pi}}} + O\left(\frac{1}{k^4 w^4}\right)$$

Absorption $$\frac{P_{abs}}{P_{in}} = \frac{P_{4\pi}}{P_{in}} = \operatorname{Im}(2\beta)$$

$$\beta(\omega) = \frac{6kc^3}{W^2} \frac{\Gamma/\omega_0^2}{\omega_0^2 - \omega^2 - i\Gamma\frac{\omega^3}{\omega_0^3}} \qquad |\underbrace{\omega - \omega_0}_{\Delta}| \ll \omega, \omega_0$$

$$= -\frac{6kc^3}{w^2\omega_0^3} \frac{\Gamma}{2\Delta + i\Gamma} = \eta_{fs} \cdot \left(\frac{i\Gamma^2}{\Gamma^2 + 4\Delta^2} - \frac{2\Delta\Gamma}{\Gamma^2 + 4\Delta^2}\right)$$

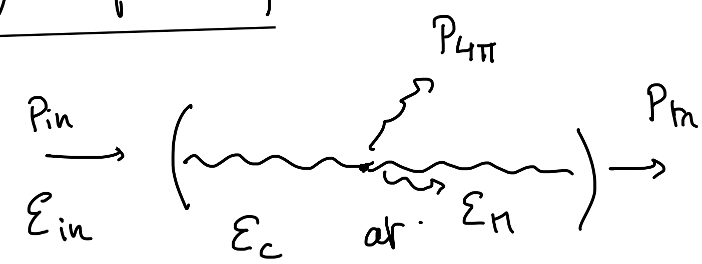
Phase shifts: $$\mathcal{E} + \mathcal{E}_H = (1 + i\beta)\mathcal{E} \sim e^{i\beta} \cdot \mathcal{E}$$




$$\phi \sim \operatorname{Re}\beta$$

$$= \eta_{fs} \frac{-2\Gamma\Delta}{\Gamma^2 + 4\Delta^2}$$

4) Cavity cooperativity




$\mathcal{E}_c$ : forward going amplitude

the atom sees $2\mathcal{E}_c$

emission into mode M : $\quad 2\mathcal{E}_M = 2 \cdot i\beta\,(2\mathcal{E}_c)$
$$= 4 i\beta\,\mathcal{E}_c$$

Steady state field: $\quad \mathcal{E}_c = iq\,\mathcal{E}_{in} + \underbrace{2\mathcal{E}_M} + r^2 e^{2ikL}\,\mathcal{E}_c$

$\vdots$

$$\mathcal{E}_c = iq\,\mathcal{E}_{in} + 4i\beta\,\mathcal{E}_c + \left(1 - q^2 + 2iq^2\frac{\delta}{\kappa}\right)\mathcal{E}_c$$

$k = \left(k_0 + \frac{\delta}{c}\right)$

$kL = k_0 L + \frac{\delta L}{c} \qquad k_0 L = 2\pi \cdot N$ defines $\delta$

$\kappa = \frac{c q^2}{L}$ cavity linewidth

$$\mathcal{E}_c = \frac{i\,\mathcal{E}_{in}}{q}\left(1 - i\,\frac{2\delta}{\kappa} - i\,\frac{4\beta}{q^2}\right)^{-1}$$

Transmission: $\quad P_{tr} = \frac{q^2|\mathcal{E}_c|^2}{2} \qquad P_{in} = \frac{|\mathcal{E}_{in}|^2}{2}$

$$\frac{P_{tr}}{P_{in}} = \frac{1}{\left(1 + \frac{\mathrm{Im}\,4\beta}{q^2}\right)^2 + \left(\frac{2\delta}{\kappa} + \frac{\mathrm{Re}\,4\beta}{q^2}\right)^2}$$

$$\mathrm{Re}\,\beta = -\eta_{fs} \cdot \frac{2\Delta\Gamma}{4\Delta^2+\Gamma^2}$$

$$\mathrm{Im}\,\beta = +\eta_{fs} \frac{\Gamma^2}{4\Delta^2+\Gamma^2}$$

$$\frac{4\beta}{g^2} = \eta\left( \frac{-2\Delta\Gamma}{\Gamma^2+4\Delta^2} + \frac{i\,\Gamma^2}{\Gamma^2+4\Delta^2} \right)$$

$$\boxed{\eta = \frac{4\eta_{fs}}{g^2} = \frac{24\,F}{\pi\, k^2 w^2}}$$

$F$ : Finesse of the cavity

Pabs : $$\frac{P_{4\pi}}{P_{in}} = \frac{8\,\mathrm{Im}\,\beta}{g^2} \frac{1}{(\quad)^2+(\quad)^2}$$

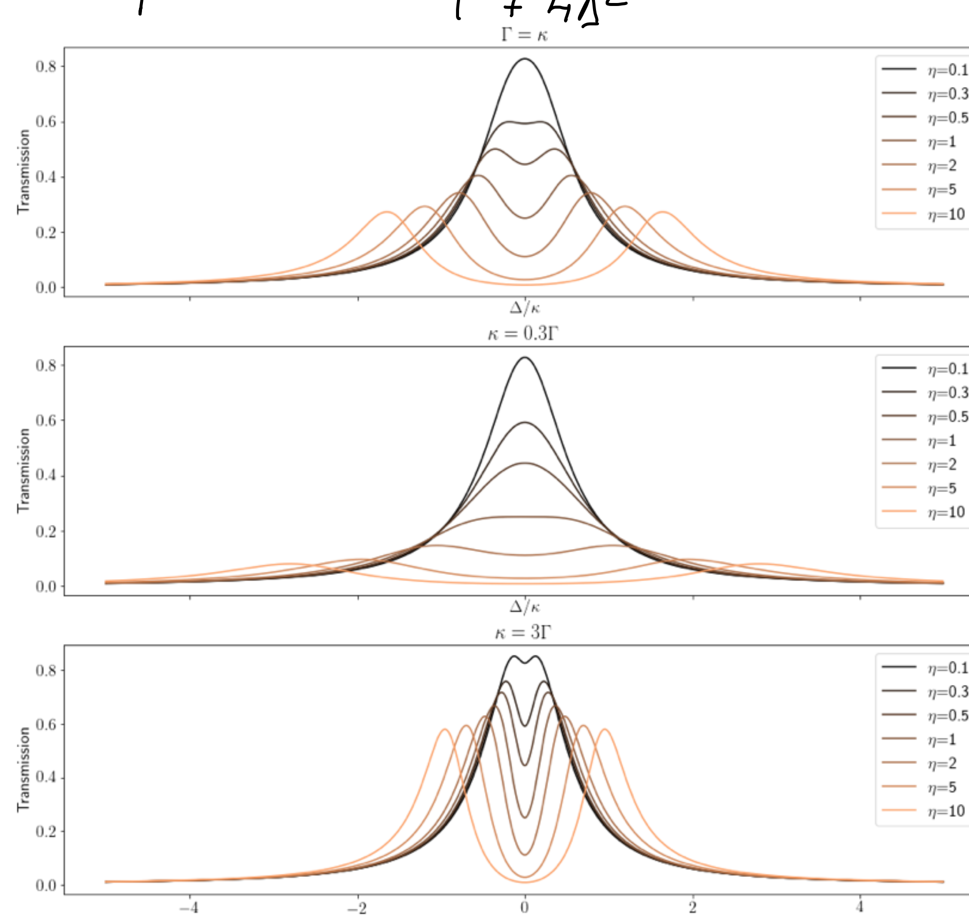
$$\frac{P_{abs}}{P_{in}} = 2\eta \frac{\Gamma^2}{4\Delta^2+\Gamma^2} \qquad \delta = 0$$

Recipie: $\eta_{fs} \to \eta$

Phase shift: $$\frac{\delta\omega_c}{\kappa} = \mathrm{Re}\left(\frac{2\beta}{g^2}\right) = -\eta \frac{\Gamma\Delta}{\Gamma^2+4\Delta^2}$$

5) Rabi Splitting:

$$\delta = \Delta$$

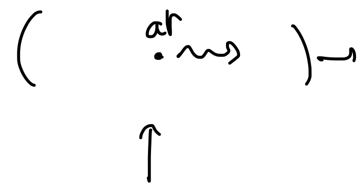



distance between peaks:

$$2g = \sqrt{\eta\kappa\Gamma}$$

$$\eta = \frac{4g^2}{\kappa\Gamma}$$

## 6) Scattering into the cavity



$$\frac{P_c}{P_{4\pi}} = \eta \cdot \frac{\kappa^2}{\kappa^2 + 4\delta^2}$$

Purcell effect

# II - Quantum problem

## 1) Jaynes-Cummings model

$|\omega_c - \omega_{ar}| \ll \omega_c, \omega_{ar}$

$\omega_{ar}$ : $|e\rangle$, $|g\rangle$

$$\hat{H}_0 = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\omega_{ar} |e\rangle\langle e|$$
$$+ \hat{H}_{int}$$

$$\hat{H}_{int} = -\hat{\vec{d}} \cdot \hat{\vec{E}}$$

$$= -d_0 \left( |e\rangle\langle g| + |g\rangle\langle e| \right) \otimes -i\,\xi_0 \left( \hat{a} - \hat{a}^\dagger \right)$$

$$\xi_0 = \sqrt{\frac{\hbar\omega_c}{2\varepsilon_0 V}}$$ V: volume of the cavity mode

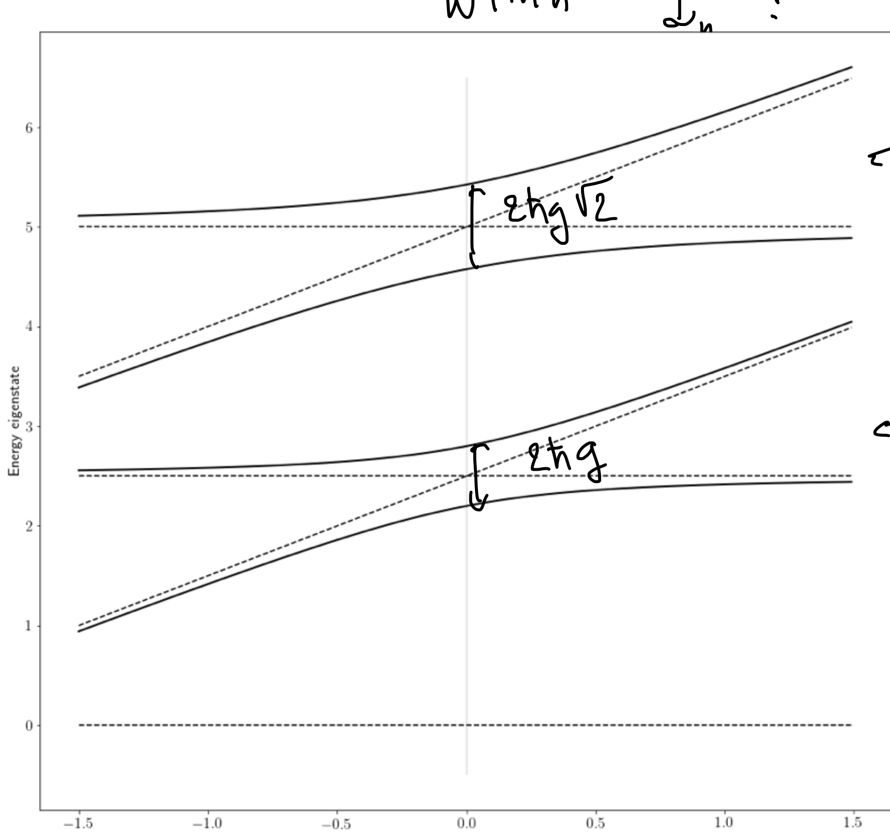
$$\hat{H}_{int} = \hbar g \left( |e\rangle\langle g|\,\hat{a} + |g\rangle\langle e|\,\hat{a}^\dagger \right)$$ (Rotating wave approximation)

Observation: $\hat{I} = |e\rangle\langle e| + \hat{a}^\dagger \hat{a}$ is a conserved quantity

Eigenspaces $\begin{cases} \mathcal{I}_n = \mathrm{Sp}\{ |e, n-1\rangle, |g, n\rangle \} & \text{dimension } 2 \\ \mathcal{I}_0 = \mathrm{Sp}\{ |g, 0\rangle \} \end{cases}$

within $\mathcal{I}_n$ :

$$\hat{H} = \begin{pmatrix} n\hbar\omega_c & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & (n-1)\hbar\omega_c + \hbar\omega_{ar} \end{pmatrix}$$




n=2

n=1

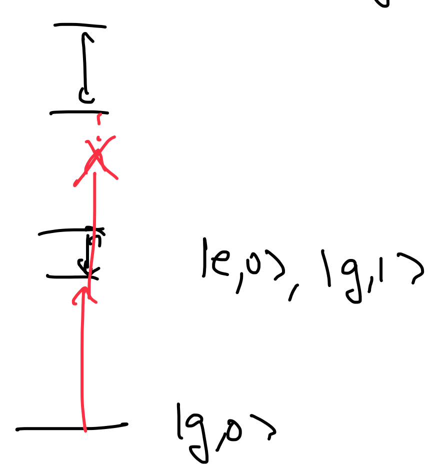
if we probe the atom + cavity system with weak probe

↳ observe anticrossing with splitting $2\hbar g$

Spectrum is non-linear! splitting $g\sqrt{n}$

- linewidth $\kappa, \Gamma \ll g$ → photon blockade.

(Kimble 2005)

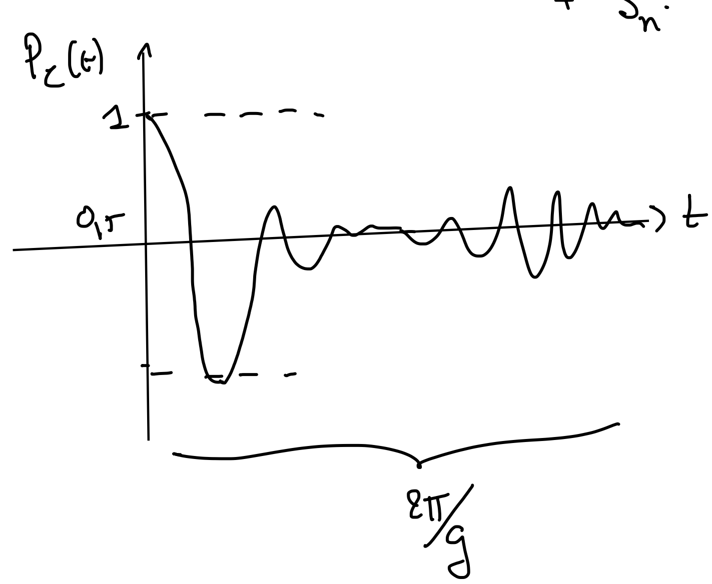



- Collapse and revivals

$$|\alpha\rangle = \sum_n e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$|\Psi_e(t)\rangle = \sum_n c_n \cos\left(g\sqrt{n+1}\cdot t\right) |e n\rangle$$

$$+ S_n \cdot \sin\left(g\sqrt{n+1}\,t\right) \cdot |g\ n+1\rangle$$




(M. Brune .. 1996)

Remark: Rabi oscillations with one photon

$\leftrightarrow$ $\langle P_{ee} \rangle \sim 1/2$ already for one photon

Condition for observing Rabi oscillations:

$$g \gg \Gamma, \kappa$$

$$\Rightarrow \quad \eta = \frac{4g^2}{\kappa \Gamma} \gg 1$$

2) Dispersive regime

$$\Delta = \omega_c - \omega_{at} \gg g$$

expansion in powers $g/\Delta$

$$\hat{H}_{JC} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\omega_{at} |e\rangle\langle e| + \hbar g \left( |e\rangle\langle g| \hat{a} + hc \right)$$

$$\hookrightarrow \quad \hat{H}_{dispersive} = g^2/\Delta \left( |e\rangle\langle e| + \frac{|e\rangle\langle e| - |g\rangle\langle g|}{2} \hat{a}^\dagger \hat{a} \right)$$

$$\hat{H}_0 = \omega_c a^\dagger a + \omega_{at} |e\rangle\langle e| \xrightarrow[\text{to field}]{\text{coupling}} \omega_c \rightarrow \omega_c + \frac{g^2}{\Delta}\left( \frac{|e\rangle\langle e| - |g\rangle\langle g|}{2} \right)$$

Conditional resonance of the cavity:

field can enter if atom in state $|e\rangle$

is blocked $|g\rangle$

$$|\psi\rangle = |0\rangle \otimes \left( \frac{|e\rangle + |g\rangle}{\sqrt{2}} \right) \xrightarrow{\text{drive}} \frac{|\alpha\rangle \otimes |e\rangle + |0\rangle \otimes |g\rangle}{\sqrt{2}}$$

$$\pi/2 \downarrow \quad \frac{|\alpha e\rangle + |\alpha g\rangle + |0 g\rangle - |0 e\rangle}{2}$$

if atom is measured in $|g\rangle$: $\frac{|\alpha\rangle + |0\rangle}{\sqrt{2}}$ Schrödinger cat

# IV. Collective effects

## 1) Classical case

free space : $\eta_{fs} \longrightarrow N\,\eta_{fs}$ , N : atom number

cavity : $\eta \longrightarrow N\,\eta\,\underbrace{\mathcal{H}}_{\downarrow}$


$$\frac{1}{N}\cdot\sum_i \cos^2 kz$$

## 2) Quantum case:

$$\hat{H} = \omega_c \hat{a}^{\dagger}\hat{a} + \sum_i |e\rangle\langle e|^i\,\omega_{at} + g\sum_j\left(|e\rangle\langle g|^j\,\hat{a} + hc\right)$$

$$|e\rangle\langle e|^i = \frac{1+\hat{\sigma}_z^{(i)}}{2}$$

$$|e\rangle\langle g|^i = \hat{\sigma}_+^{(i)}$$

"large spin" operators: $\hat{J}_{x,y,z} = \frac{1}{2}\sum_j \hat{\sigma}_{x,y,z}^{(j)} \qquad \hat{J}_{\pm} = \sum_j \hat{\sigma}_{\pm}^{(j)}$

N spin $1/2$ $\Longrightarrow$ 1 spin $N/2$

$$\hat{H} = \omega_c \hat{a}^{\dagger}\hat{a} + \left(\frac{N}{2}+\hat{J}_z\right)\hat{\omega}_{at} + g\left(\hat{a}\,\hat{J}_+ + hc\right)$$

Tavis-Cummings model

Structure of the eigenstats of $\hat{J}$ : $|J, M\rangle$ (angular momentum algebra)

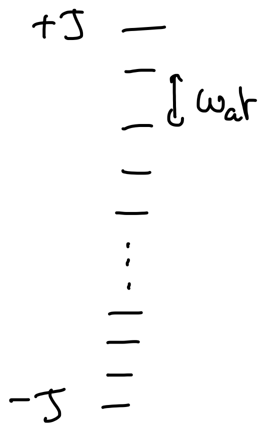
the state with maximum M is $\underbrace{|g, g, \cdots, g\rangle}_{\text{all atoms in state } |g\rangle} = \left|\frac{N}{2}, -\frac{N}{2}\right\rangle$

Generate the other eigenstats by applying $\hat{J}_+$ :

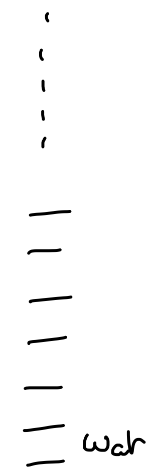
$$\hat{J}_+|J\,M\rangle = \sqrt{(J+M+1)(J-M)}\cdot|J\;M+1\rangle$$

In contrast with a two level system, we have $\hat{J}_+^2 \neq 0$

## Spectrum of the atomic part:




in the weak excitation regime, ($n_{ph} < N$), we can map this onto a harmonic oscillator:
(as long as we don't come too close to +J ...)




Holstein - Primakoff transformation:

$$\hat{J}_+ = \sqrt{N}\cdot \hat{b}^+ \left(1 - \frac{\hat{b}^+\hat{b}}{N}\right)^{1/2} \sim \hat{b}^+\sqrt{N}$$

$$\hat{J}_- = \sqrt{N}\left(1 - \frac{\hat{b}^+\hat{b}}{N}\right)^{1/2} \hat{b} \sim \hat{b}\sqrt{N}$$

$$\hat{J}_z = \hat{b}^+\hat{b} - N/2$$

and $[\hat{b}, \hat{b}^+] = 1 \iff [\hat{J}_+, \hat{J}_-] = 2\hat{J}_z$

Then :

$$\hat{H} \sim \omega_c \hat{a}^+\hat{a} + \omega_{at}\hat{b}^+\hat{b} + g\sqrt{N}\left(\hat{a}^+\hat{b} + \hat{b}^+\hat{a}\right)$$

coupled harmonic oscillator model!

→ eigenmodes are harmonic oscillators, $\begin{cases} \hat{C}_+ = \cos\theta/2\ \hat{b} + \sin\frac{\theta}{2}\hat{a} \\ \hat{C}_- = -\sin\theta/2\ \hat{b} + \cos\frac{\theta}{2}\hat{a} \end{cases}$

- The spectrum <u>is linear again</u>! harmonic oscillators $\hat{C}_+, \hat{C}_-$
- The excitations are called <u>Polaritons</u>