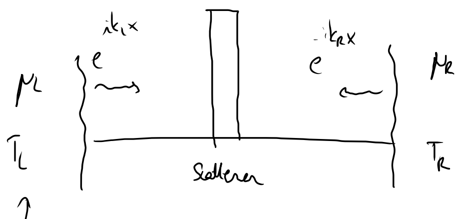
Mesoscopic conductors

1D conductor



reservoirs: states filled according to f_L

$$f_i = \frac{1}{e^{(E-\mu_i)/k_B T_i} + 1}$$

(H) $E(k) = \frac{\hbar^2 k^2}{2m}$

Right moving e⁻:

$$\hat{\Psi}_L(x,t) = \frac{1}{\sqrt{2\pi}} \int dk\, \hat{a}_L(k)\, e^{i(kx - Et/\hbar)}$$

$$\{\hat{a}_\alpha(k), \hat{a}^+_\beta(k')\} = \delta_{\alpha\beta}\, \delta(k-k)$$

$$\langle \hat{a}^+_\alpha(k') \hat{a}_\beta(k) \rangle = \delta_{\alpha\beta}\, \delta(k-k')\, f_\alpha(E_k)$$

We rewrite this state as a fct of energy:

$$\boxed{\Psi_L(x,t) = \frac{1}{\sqrt{2\pi}} \int dE\, \frac{\hat{a}_L(E)}{\sqrt{\hbar v(E)}}\, e^{i(kx - Et/\hbar)}}$$

$v(E)$: electron velocity $= \frac{1}{\hbar}\frac{\partial E}{\partial k} = \frac{\hbar k}{m}$

$$\{\hat{a}_L(E), \hat{a}^+_L(E')\} = \delta(E-E')$$

$$\langle \hat{a}^+_\alpha(E') \hat{a}_\beta(E) \rangle = \delta_{\alpha\beta}\, \delta(E-E')\, f_\alpha(E)$$

What is the expression of the electrical current operator in 1D

$\hat{I}$ satisfies the conservation equation:

$$e\frac{\partial \Psi^+\Psi}{\partial t} = -\frac{\partial}{\partial x}\hat{I} \qquad e\hat{\Psi}^+\hat{\Psi}: \text{local charge density}$$

Using Schrödinger equation:

$$i\hbar \partial_t \hat{\Psi} = H\hat{\Psi} \qquad H = \frac{\vec{p}^2}{2m} + U(x) = -\frac{\hbar^2\partial_x^2}{2m} + U(x)$$

$$-i\hbar \partial_t \hat{\Psi}^+ = H\hat{\Psi}^+$$

$$\rightarrow e\frac{\hbar}{2m}\partial_x\left((\partial_x\Psi^+)\Psi - \Psi^+(\partial_x\Psi)\right) \overset{!}{=} -\partial_x\hat{I}$$

$$\rightarrow \boxed{\hat{I} = e\frac{\hbar}{2mi}\left[\Psi^+(\partial_x\Psi) - (\partial_x\Psi^+)\Psi\right]}$$

We now compute the current operator on the left using $\hat{\Psi}_L(x,t)$.

$$k = \frac{mv}{\hbar}$$

$$\hat{I}_L(x,t) = \frac{e}{h}\int dE \int dE'\, a^+_L(E')\, a_L(E)\, \frac{v(E)+v(E')}{2\sqrt{v(E)v(E')}} \cdot$$

$$\cdot\, e^{i(k(E)-k(E'))x}\, e^{-i(E-E')t/\hbar}$$

$\langle \hat{I}_L \rangle$

$$\rightarrow \langle a^+_L(E')\, a_L(E) \rangle = \delta(E-E')\, f_L(E)$$

$$\boxed{\langle \hat{I}_L \rangle = \frac{e}{h}\int dE\, f_L(E)}$$

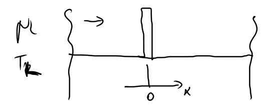
if I consider only fully occupied states, $f_L(E) = 1$ within the energy range dE

$$\boxed{dI_L = \frac{e}{h}\, dE}$$

$$\rightarrow dN_F = \frac{1}{h}\, dE$$

$$dN_B = \frac{N_{ph}}{h}\, d(h\nu)$$

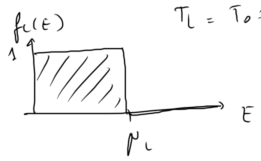
We can come back to the scattering setup



$\mu_L = \mu_0 + eV$
$\mu_R = \mu_0$

$T_R = T_0 = 0$
$T_L = T_0 + \Delta T$

$\frac{e}{h}\int dE\, f_L(E)$



$$I_L = \frac{e}{h}\int_{\mu_0}^{\mu_0+eV} dE = \frac{e^2 V}{h} = \frac{e}{h}\cdot eV$$

Reminder for Q.M.:

$$\varphi_L(E,x) \begin{cases} = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\hbar v_L(E)}}\left(e^{ik_L(E)x} + r(E)\, e^{-ik_L(E)x}\right) & x<0 \\ = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\hbar v_R(E)}}\, t(E)\, e^{ik_R(E)x} & x>0 \end{cases}$$

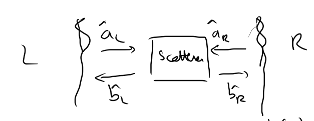
$$\varphi_R(E,x) = \begin{cases} & x<0 \\ & x>0 \end{cases}$$

$$S = \begin{pmatrix} r(E) & t'(E) \\ t(E) & r'(E) \end{pmatrix}$$

$\hat{a}_L(E), \hat{a}_R(E)$: act onto 2 separate Fock spaces that define the occupation number of the states emitted by the left/right reservoirs.

$\hat{b}_L(E), \hat{b}_R(E)$: for the outgoing states

$$\begin{pmatrix} \hat{b}_L \\ \hat{b}_R \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} \hat{a}_L \\ a_R \end{pmatrix}$$



$$\hat{\Psi}(x,t) = \begin{cases} \int dE \frac{1}{\sqrt{2\pi\hbar v_L(E)}}\left(\hat{a}_L(E)\, e^{i(k(E)x - Et/\hbar)} + \hat{b}_L(E)\, e^{-i(k_L(E)x + Et/\hbar)}\right) & x<0 \\ \int dE \frac{1}{\sqrt{2\pi\hbar v_R(E)}}\left(\hat{b}_R(E)\, e^{i(k_R(E)x - Et/\hbar)} + \hat{a}_R(E)\, e^{-i(k_R(E)x - Et/\hbar)}\right) & x>0 \end{cases}$$

$$\Rightarrow \hat{I}_L(x,t) = \frac{e}{h}\int dE \int dE' \left(a_L^+(E')\, a_L(E) - b_L^+(E')\, b_L(E)\right) \cdot \frac{v(E)+v(E')}{2\sqrt{v(E)+v(E')}}\, e^{i(k(E)-k(E'))x}\, e^{-i(E-E')t/\hbar}$$

Approximation that is fully justified for most experiments when $|E-E'| \ll E_F$

$v(E) \sim v(E') \sim v_F$

$$\hat{I}_L(t) = \frac{e}{h}\iint dE\, dE' \left[a_L^+(E')\, a_L(E) - b_L^+(E')\, b_L(E)\right] e^{-i(E-E')t/\hbar} \qquad x<0$$

$$\hat{I}_R(t) = -\frac{e}{h}\iint dE\, dE' \left[a_R^+(E')\, a_R(E) - b_R^+(E')\, b_R(E)\right] e^{-i(E-E')t/\hbar} \qquad x>0$$

$\rightarrow$ these expressions allow us to derive the well-know Landauer formula

$$\langle \hat{I}_R \rangle = -\frac{e}{h}\iint dE\, dE' \left\langle a_R^+(E')\, a_R(E) - |t|^2 a_L^+(E')\, a_L(E) - |r'|^2 a_R^+(E')\, a_R(E) \right\rangle$$

$$= -\frac{e}{h}\iint dE\, dE' \left(f_R(E)\, \delta(E-E') - |t|^2 f_L(E)\, \delta(E-E') - |r'|^2 f_R(E)\, \delta(E-E') \right)$$

$$= -\frac{e}{h}\int dE \Big[\underbrace{(1 - |r'|^2)}_{|t|^2 \text{ using } S^+S = \mathbb{1}} f_R(E) - |t|^2 f_L(E)\Big]$$

Properties of the S matrix:

$$a^+a = b^+b = (Sa)^+ Sa = a^+ \underbrace{S^+S}_{\mathbb{1}} a$$

$\rightarrow$ S matrix is unitary

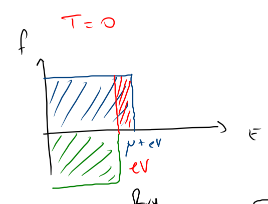
$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \rightarrow S^+S = \mathbb{1} \rightarrow \begin{cases} |r|^2 + |t|^2 = 1 \\ |r'|^2 + |t'|^2 = 1 \end{cases}$$

$$\begin{cases} |r|^2 + |t'|^2 = 1 \\ |r'|^2 + |t|^2 = 1 \end{cases}$$

$$\boxed{\langle \hat{I}_R \rangle = \frac{e}{h}\int dE\, |t|^2 \left(f_L(E) - f_R(E)\right)}$$

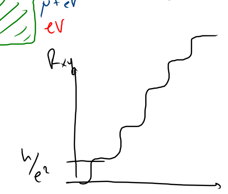
Landauer formula

T = 0



$$= \frac{e}{h} |t|^2 \cdot eV = \frac{e^2 V}{h} \cdot T$$

$I = G\, V$ with $G = \frac{e^2 T}{h}$



$T = 1$ $\quad I = \frac{e^2}{h} V$ $\qquad R_0 = \frac{h}{e^2}$

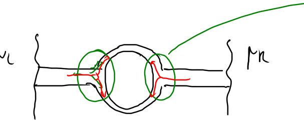
$R = \frac{h}{e^2}$ $\qquad G_0 = \frac{e^2}{h}$

Lecture 2:

$G = \frac{e^2}{h} \cdot T$

Aharonov - Bohm ring

T - junctions



$\vec{B} \otimes$ μ_L μ_R

dynamical phase:

$\chi_i = e^{ik(E)x}$

→ an e- that follows the time-reversed path picks up the same phase

geometrical phase:

Aharonov - Bohm phase

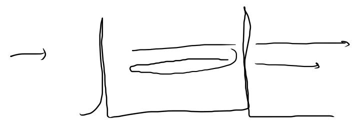
vector potential $\vec{A}$ s.t. $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\varphi_i = \frac{e}{h} \int_{C_i} \vec{A} \cdot d\vec{l}$$

$$\rightarrow \varphi_u - \varphi_d = \frac{e}{h} \oint_C \vec{A} \cdot d\vec{l} \underset{\text{Stokes}}{=} \frac{e}{h} \int_S \vec{B} \cdot d\vec{S} = 2\pi \Phi / \Phi_0$$

$\Phi_0 = h/e$: flux quantum

→ opposite phase shift if time-reversed path.



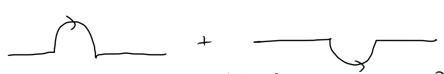
t_1 t_2

$t_1 t_2$

$t_1 r_2 r_1 t_2$

$$\left(T = \frac{T_L T_R}{1 - T_L T_R} . \right)$$

AB - ring: different types of geometries

① 

$$t = \frac{1}{2} \left(e^{i(\varphi_1 + \chi_1)} + e^{i(\varphi_2 + \chi_2)} \right)$$

$$T = |t|^2 = \dots = 1 + \cos(2\pi\phi/\phi_0) \cos(\chi_1 - \chi_2) - \underbrace{\sin(2\pi\phi/\phi_0)}_{\text{odd}} \sin(\chi_1 - \chi_2) .$$

↳ violates Onsager symmetries !

Aside 1: Onsager symmetries

Q.M. micro-reversibility:

$$i\hbar \partial_t \psi(t) = H_B \psi(t)$$

$$i\hbar \partial_t \psi^*(t) = H_{-B} \psi^*(t)$$

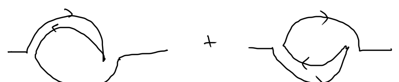
→ imposes additional conditions on the S-matrix

$$c_{out} = S_B \, c_{in} \Rightarrow c_{out}^{new} = S_{-B} \, c_{in}^{new}$$

$c_{out}^{new} = c_{in}^*$, $c_{in}^{new} = \underbrace{c_{out}^*}_{S_B^* c_{in}^*}$

$$\Leftrightarrow c_{in}^* = \underbrace{S_{-B} S_B^*}_{\stackrel{!}{=} \mathbb{1}} c_{in}^*$$

$$\rightarrow \boxed{S_B = S_{-B}^T} \Rightarrow T(B) = T(-B)$$

② 

$$t \approx \frac{1}{2} e^{i(\chi_1 + \varphi_1)} \frac{1}{2} e^{i(\chi_1 - \varphi_1)} \frac{1}{2} e^{i(\chi_2 + \varphi_2)}$$

$$+ \frac{1}{2} e^{i(\chi_2 + \varphi_2)} \frac{1}{2} e^{i(\chi_2 - \varphi_2)} \frac{1}{2} e^{i(\chi_1 + \varphi_1)}$$

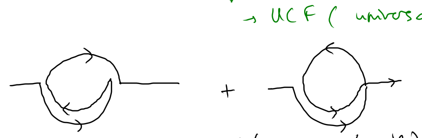
$$= \frac{1}{8} \left(e^{i(2\chi_1 + \chi_2 + \varphi_2)} + e^{i(2\chi_2 + \chi_1 + \varphi_1)} \right)$$

$$T = |t|^2 \sim 1 + \cos(\chi_1 - \chi_2) \cos 2\pi\phi/\phi_0 + \sin(\chi_1 - \chi_2) \sin 2\pi\phi/\phi_0$$

↑ cancels the problematic term from before

① + ② : $T \sim \underbrace{\cos(\chi_1 - \chi_2)}\cos(2\pi\phi/\phi_0)$

→ UCF (universal conductance fluct.)

③ 

$$t \sim \frac{1}{8} \left(e^{i(\chi_2 + \varphi_2)} e^{i(\chi_2 - \varphi_2)} e^{i(\varphi_1 + \chi_1)} + e^{i(\chi_2 + \varphi_2)} e^{i(\chi_1 - \varphi_1)} e^{i(\varphi_2 + \chi_2)} \right)$$

$$\Rightarrow T \sim \cos^2(2\pi\phi/\phi_0) \sim \frac{1}{2} \left(1 + \cos(2 \cdot 2\pi\phi/\phi_0) \right)$$

→ does not depend on the dynamical phases

→ it oscillates with $\phi_0/2$

→ weak-localization contributions

$$\vec{J} = \sigma \vec{E} \longrightarrow (\sigma + d\sigma) \vec{E}$$

↑ weak-localization term

$$\Rightarrow T = \frac{(1 - \cos\chi)(1 + \cos(2\pi\phi/\phi_0))}{\sin^2\chi + \left(\cos\chi - \frac{1 + \cos 2\pi\phi/\phi_0}{2}\right)^2}$$

Towards quantum thermodynamics through thermoelectricity

<u>Idea</u>: you generate a charge current from ∇T (thermocouple)
heat current from V

→ 1821: Thomas Johann Seebeck
→ 1834: Jean-Claude Peltier } manifestation of the same phenomenon
→ 1850: Lord Kelvin

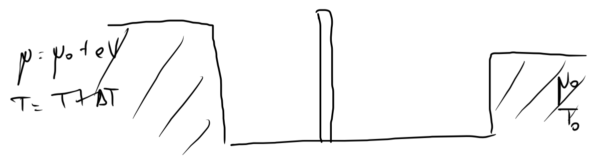
↓ 1931: Lars Onsager: established the reciprocal relations between Peltier & Seebeck coeffs.

(I) Linear response theory

$\Delta T \ll T = \frac{T_L + T_R}{2}$

$eV \ll \mu$

$\Delta T \ll \bar{T}$
$eV \ll \mu_0$.



$$I = 2\frac{e}{h}\int dE\, T(E)\left(f_L - f_R\right)$$

$$J = \frac{2}{h}\int dE\,(E-\mu)\,T(E)\left(f_L - f_R\right)$$

$$f_{\mu,T}(E) = f(\mu_0, T_0, E) + \frac{\partial f}{\partial \mu}\Big|_{\mu=\mu_0}\overbrace{(\mu-\mu_0)}^{eV} + \frac{\partial f}{\partial T}\Big|_{T=T_0}\overbrace{(T-T_0)}^{\Delta T}$$

$$\frac{\partial f}{\partial \mu} = \left(-\frac{\partial f}{\partial E}\right) \quad ; \quad \frac{\partial f}{\partial T} = \frac{E-\mu}{T}\left(-\frac{\partial f}{\partial E}\right)$$

$$\left(-\frac{\partial f}{\partial E}\right) = \frac{1}{4k_BT}\,\frac{1}{\cosh^2\left(\frac{E-\mu}{2k_BT}\right)}.$$

$$I = \frac{2e^2}{h}\int dE\, T(E)\left(-\frac{\partial f}{\partial E}\right) V + \frac{2e}{hT}\int dE\,(E-\mu)\,T(E)\left(-\frac{\partial f}{\partial E}\right)\Delta T$$

$$J = \frac{2e}{h}\int dE\,(E-\mu)\,T(E)\left(-\frac{\partial f}{\partial E}\right) V + \frac{2}{hT}\int dE\,(E-\mu)^2\,T(E)\left(-\frac{\partial f}{\partial E}\right)\Delta T$$

$$\begin{pmatrix} I \\ J \end{pmatrix} = \begin{pmatrix} G & \alpha \\ \beta & K' \end{pmatrix}\begin{pmatrix} V \\ \Delta T \end{pmatrix}$$

odd   even fct of E

$$\boxed{\alpha = \frac{2e}{hT}\int dE\,(E-\mu)\,T(E)\left(-\frac{\partial f}{\partial E}\right)}$$

$$G = \frac{2e^2}{h}\int dE\, T(E)\left(-\frac{\partial f}{\partial E}\right)$$

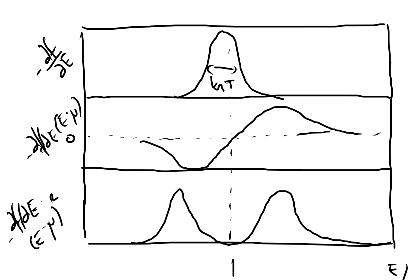
$$K' = \frac{2}{hT}\int dE\,(E-\mu)^2\,T(E)\left(-\frac{\partial f}{\partial E}\right)$$

$\beta = \alpha T$ ← Kelvin's law

$T(E) = T$

i) Kelvin's law

ii)



Sommerfeld expansion: $T(E) \simeq T(\mu) + \frac{\partial T}{\partial E}(E-\mu)$

$$\frac{K'}{T} = \dots = \frac{2k_B^2\pi^2}{3h}\,T(\mu)$$

$$G = \frac{2e^2}{h}\,T(\mu)$$

} $\frac{K'}{GT} = \frac{k_B^2\pi^2}{3e^2} = L_0$ Lorentz number

Wiedemann-Franz law

$K' = L_0\, G\, T$

Equilibrium thermodynamics:

infinitesimal change in free energy:

$$\delta U = T\,dS + \mu\,dN + (p\,dV)$$

S, N, V : ext. quantities → scale with syst. size

T, μ, P : intensive

Energy flux

$E/(s \cdot A)$: $J_u = T J_S + \mu J_N$

How does the system respond to a spatial variation of temperature or chemical potential?

$$\nabla \cdot J_u = (\nabla T) \cdot J_S + T \cdot (\nabla J_S) + \nabla \mu \cdot J_N + \mu \nabla J_N$$

- 1st law of thermodynamics : $\nabla J_u = 0$
- # of particles is conserved : $\nabla \cdot J_N = 0$
- Def : $\nabla \cdot J_S = \dot{S}$: entropy productio rate

$$\Rightarrow \quad 0 = J_S \nabla T + T \cdot \dot{s} + \nabla \mu \, J_N$$

if $\dot{s} = 0$: $\nabla \mu \, J_N = - J_S \cdot \nabla T$

$\Leftrightarrow \quad eV \, J_N = - J_n \cdot T \cdot \Delta T$

$\Leftrightarrow \quad I V = - J \, T \, \Delta T$

Each syst. that conserves energy and # of particles will give rise to thermo-electric effects.